

# WELLPPOSEDNESS FOR DENSITY-DEPENDENT INCOMPRESSIBLE VISCOUS FLUIDS ON THE TORUS $\mathbb{T}^3$

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ABSTRACT. We investigate the local wellposedness of incompressible inhomogeneous Navier-Stokes equations on the Torus  $\mathbb{T}^3$ , with initial data in the critical Besov spaces. Under some smallness assumption on the velocity in the critical space  $B_{2,1}^{\frac{1}{2}}(\mathbb{T}^3)$ , the global-in-time existence of the solution is proved. The initial density is required to belong to  $B_{2,1}^{\frac{3}{2}}(\mathbb{T}^3)$  but not supposed to be small.

## 1. INTRODUCTION AND MAINS STATEMENTS

Incompressible flows are often modeled by the incompressible homogeneous Navier-Stokes system (1), e.g the density of the fluid is supposed to be a constant

$$(1) \quad \begin{cases} \partial_t v + v \cdot \nabla v - \Delta v &= -\nabla p \\ \operatorname{div} v &= 0 \\ v|_{t=0} &= v_0. \end{cases}$$

However, this model is sometimes far away from the physical situation. Concerning models of blood and rivers, even if the fluid is incompressible, its density can not be considered constant, owing to the complexity of the structure of the flow. As a result, a model which takes into account such constraints, has to be considered. That is the so-called Inhomogeneous Navier-Stokes system, given by

$$(2) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) &= 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \Delta u + \nabla \Pi &= 0 \\ \operatorname{div} u &= 0 \\ (\rho, u)|_{t=0} &= (\rho_0, u_0). \end{cases}$$

which is equivalent to the system below, by virtue of the transport equation

$$(3) \quad \begin{cases} \partial_t \rho + u \cdot \nabla \rho &= 0 \\ \rho(\partial_t u + u \cdot \nabla u) - \Delta u + \nabla \Pi &= 0 \\ \operatorname{div} u &= 0 \\ (\rho, u)|_{t=0} &= (\rho_0, u_0), \end{cases}$$

where  $\rho = \rho(t, x) \in \mathbb{R}^+$  stands for the density and  $u = u(t, x) \in \mathbb{T}^3$  for the velocity field. The term  $\nabla \Pi$  (namely the gradient of the pressure) may be seen as the Lagrangian multiplier associated with the constraint  $\operatorname{div} u = 0$ . The initial data  $(\rho_0, u_0)$  are prescribed. Notice, we choose the viscosity of the fluid equal to 1, in a sake of simplicity.

Let us recall some well-known results about the two above systems (homogeneous versus inhomogeneous). In the homogeneous case, the celebrated theorem of J. Leray [15] proves the global existence of weak solutions with finite energy in any space dimension. The uniqueness is guaranteed in dimension 2, whereas in dimension 3, this is still an open question. In deal with this issue, H. Fujita and T. Kato [10] built some global strong solutions in the context of scaling invariance spaces, namely spaces which have the same scaling as the system (1). Such spaces are said to be critical, in the sense that their norm is invariant for any  $\lambda > 0$  under the transformation

$$v_0(x) \mapsto \lambda v_0(\lambda x) \quad \text{and} \quad v(t, x) \mapsto \lambda v(\lambda^2 t, \lambda x).$$

The point is that such solutions are unique in this framework. In the inhomogeneous case, Leray's approach is still relevant for the system (2). Indeed, if the initial density  $\rho_0$  is non negative and belongs to  $L^\infty$  and if  $\sqrt{\rho_0} u_0$  belongs to  $L^2$ , then there exists some global weak solutions  $(\rho, u)$  with finite energy. However, the question of uniqueness has not been solved, even in dimension 2. We refer the reader to the paper of A. Kazhikhov [12], J. Simon [18] for the existence of global weak solutions. The unique resolvability of (2) is first established by the works of O. Ladyzenskaja and V. Solonnikov [13] in the case of a bounded domain  $\Omega$  with homogeneous Dirichlet condition for the velocity  $u$ . As one has already mentioned previously, the approach initiated by H. Fujita and T. Kato is particularly efficient in the scaling invariance framework to face the uniqueness problem. A natural question is to wonder if such an approach is relevant for incompressible inhomogeneous fluids. If one believes so, scaling considerations should help us to find an adapted functional framework. Firstly, one can check that (3) is invariant under the scaling transformation (for any  $\lambda > 0$ )

$$(\rho_0, u_0)(x) \mapsto (\rho_0, \lambda u_0)(\lambda x) \quad \text{and} \quad (\rho, u, \Pi)(t, x) \mapsto (\rho, \lambda u, \lambda^2 \Pi)(\lambda^2 t, \lambda x).$$

That is an easy exercise to check that  $\dot{B}_{2,1}^{\frac{3}{2}}(\mathbb{R}^3) \times \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^3)$  is scaling invariant under this transformation, in dimension 3, e.g

$$\|\rho_0(\lambda x)\|_{\dot{B}_{2,1}^{\frac{3}{2}}(\mathbb{R}^3)} = \|\rho_0\|_{\dot{B}_{2,1}^{\frac{3}{2}}(\mathbb{R}^3)} \quad \text{and} \quad \|\lambda u_0(\lambda x)\|_{\dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^3)} = \|u_0\|_{\dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^3)}.$$

Secondly, as the system (3) degenerates if  $\rho$  vanishes or becomes unbounded, we further assume that the density is away from zero ( $\rho_0^{\pm 1} \in L^\infty$ ). Denoting

$$\frac{1}{\rho_0} \stackrel{\text{def}}{=} 1 + a_0 \quad \text{and} \quad \frac{1}{\rho} \stackrel{\text{def}}{=} 1 + a,$$

the incompressible inhomogeneous Navier-Stokes system (3) can be rewritten as

$$(4) \quad \begin{cases} \partial_t a + u \cdot \nabla a & = & 0 \\ \partial_t u + u \cdot \nabla u + (1 + a)(\nabla \Pi - \Delta u) & = & 0 \\ \operatorname{div} u & = & 0 \\ (a, u)|_{t=0} & = & (a_0, u_0), \end{cases}$$

The question of unique solvability of the above system (4) has been addressed by many authors. Let us highlight the work of R. Danchin [6], who studied the unique solvability of (4) with constant viscosity coefficient and in scaling invariant (e.g critical) Besov spaces in the whole space  $\mathbb{R}^N$ . This generalized the celebrated results by H. Fujita and T. Kato, devoted to the classical homogeneous Navier-Stokes system (1). Indeed, R. Danchin proved in [6] (under the assumption the density is close to a constant) a local well-posedness for large initial velocity and a global well-posedness for initial velocity small with respect to the viscosity. More precisely, he proved that if the initial data  $(a_0, u_0)$  belongs to  $\dot{B}_{2,\infty}^{\frac{N}{2}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \times \dot{B}_{2,1}^{\frac{N}{2}-1}(\mathbb{R}^N)$ , with  $a_0$  small enough in  $\dot{B}_{2,\infty}^{\frac{N}{2}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ , then the system (4) has a unique local-in-time solution. In addition, assuming the velocity  $u_0$  is also small enough in the space  $\dot{B}_{2,1}^{\frac{N}{2}-1}(\mathbb{R}^N)$ , the solution is global.

Our main motivation in this paper is to investigate the local and global wellposedness of the incompressible inhomogeneous Navier-Stokes system, in the case of critical Besov spaces and on the torus  $\mathbb{T}^3$ . The aim is to get rid of the smallness condition on the density, and just keeping the smallness one on the initial velocity. We point out that such a result has been already proved in the whole space  $\mathbb{R}^3$ . We refer the reader to the paper [4] of H. Abidi, G. Gui and P. Zhang. The main difference between their work and ours is that, on the torus, we have to be careful, owing to the average of the velocity  $u$ , which is not preserved, contrary to the case of classical Navier-Stokes system (1). As a consequence, a lot of "classical results" such as Gagliardo-Nirenberg inequalities and Sobolev embeddings, have to take into account the average of the velocity  $u$ . We will collect them in section 2. Let us give some

remarks about this.

**Notation** In the sequel, we shall denote by

$$\bar{m} \stackrel{\text{def}}{=} \int_{\mathbb{T}^3} m(x) dx, \quad \text{where } |\mathbb{T}^3| = 1.$$

**Remark 1.1.** *It is clear that  $\bar{\rho} = \bar{\rho}_0$ . Indeed, an integration on the mass conservation equation combining with the fact  $\int_{\mathbb{T}^3} u \cdot \nabla \rho = 0$  gives*

$$\int_{\mathbb{T}^3} \rho(t, x) dx = \int_{\mathbb{T}^3} \rho_0(x) dx.$$

Notice that by virtue of the divergence free condition on the velocity  $u$ , the average of any function of  $\rho$  is preserved. In particular, the average of  $a$  is conserved.

**Remark 1.2.** *An integration on the momentum equation of the system (2) (the terms  $\int_{\mathbb{T}^3} \text{div}(\rho u \otimes u)$ ,  $\int_{\mathbb{T}^3} \Delta u$  and  $\int_{\mathbb{T}^3} \nabla \Pi$  are nul) implies*

$$\int_{\mathbb{T}^3} (\rho u)(t, x) dx = \int_{\mathbb{T}^3} \rho_0 u_0(x) dx.$$

**Remark 1.3.** *Notice that  $\rho - \bar{\rho}$  is also solution of the transport equation. Thus, if we take the  $L^2$  inner product of this mass conservation equation with  $\rho - \bar{\rho}$  itself, we get the energy conservation of the quantity  $\|\rho - \bar{\rho}\|_{L^2}$ , because of divergence-free condition of  $u$ . Therefore we have :*

$$\|\rho - \bar{\rho}\|_{L^2} = \|\rho_0 - \bar{\rho}_0\|_{L^2}.$$

In this paper, our main Theorem can be stated as follows

**Theorem 1.1** (Main theorem). *Let  $a_0 \in B_{2,1}^{\frac{3}{2}}$ ,  $u_0 \in B_{2,1}^{\frac{1}{2}}$ , such that*

$$(5) \quad \text{div } u_0 = 0 \quad ; \quad 1 + a_0 \geq b \quad \text{for some positive constant } b \quad \text{and} \quad \int_{\mathbb{T}^3} \frac{1}{1 + a_0(x)} u_0(x) dx = 0.$$

*Then there exists a positive time  $T_*$  such that the system (4) has a unique local-in-time solution : for any  $T < T_*$ ,*

$$(a, u, \Pi) \in \mathcal{C}([0, T], B_{2,1}^{\frac{3}{2}}) \times \mathcal{C}([0, T], B_{2,1}^{\frac{1}{2}}) \cap L^1([0, T], B_{2,1}^{\frac{5}{2}}) \times L^1([0, T], B_{2,1}^{\frac{1}{2}}).$$

*In addition, there exists a constant  $c$  (depending on  $\|a_0\|_{B_{2,1}^{\frac{3}{2}}}$ ) such that*

$$\text{if } \|u_0\|_{B_{2,1}^{\frac{1}{2}}} \leq c, \quad \text{then } T_* = +\infty.$$

Our main Theorem 1.1 relies on two Theorems, given below. Indeed, we will face the question of local wellposedness and global wellposedness in a different way. The first one deals with the local wellposed issue: until a small time, we may control the velocity  $u$  in some functional Besov spaces, by the initial data  $u_0$ . It can stated as follows

**Theorem 1.2** (Local-wellposedness theorem). *Let  $a_0 \in B_{2,1}^{\frac{3}{2}}$ ,  $u_0 \in B_{2,1}^{\frac{1}{2}}$ , such that*

$$(6) \quad \text{div } u_0 = 0 \quad ; \quad 1 + a_0 \geq b \quad \text{for some positive constant } b.$$

*Then there exists a positive time  $T_*$  such that the system (4) has a unique local-in-time solution : for any  $T < T_*$ ,*

$$(a, u, \Pi) \in \mathcal{C}([0, T], B_{2,1}^{\frac{3}{2}}) \times \mathcal{C}([0, T], B_{2,1}^{\frac{1}{2}}) \cap L^1([0, T], B_{2,1}^{\frac{5}{2}}) \times L^1([0, T], B_{2,1}^{\frac{1}{2}}).$$

In addition, there exists a small constant  $c$  depending on  $\|a_0\|_{B_{2,1}^{\frac{3}{2}}}$  such that if

$$\|u_0\|_{B_{2,1}^{\frac{1}{2}}} \leq c,$$

therefore,  $T_* \geq 1$  and one has for any  $T < T_*$ ,

$$(7) \quad \text{Density estimate: } \|a\|_{L_T^\infty(B_{2,1}^{\frac{3}{2}})} \leq \|a_0\|_{B_{2,1}^{\frac{3}{2}}} \exp\left(C \|u\|_{L_T^1(B_{2,1}^{\frac{5}{2}})}\right).$$

$$(8) \quad \text{Velocity estimate: } \|u\|_{L_T^\infty(B_{2,1}^{\frac{1}{2}})} + \|u\|_{L_T^1(B_{2,1}^{\frac{5}{2}})} + \|\nabla \Pi\|_{L_T^1(B_{2,1}^{\frac{1}{2}})} \leq C \|u_0\|_{B_{2,1}^{\frac{1}{2}}}.$$

**Remark 1.4.** The difficulty, as mentionned previously, is that the density  $a$  is not supposed to be small. To overcome this issue, we split the density  $1 + a$  into

$$1 + a = (1 + S_m a) + (a - S_m a), \quad \text{where } S_m a \stackrel{\text{def}}{=} \sum_{j \leq m-1} \Delta_j a.$$

The first part is then regular enough, the second part can be made small enough, for some large enough integer  $m$ : we fix  $m$  in the sequel such that  $\|a - S_m a\|_{B_{2,1}^{\frac{3}{2}}} \leq c$ .

The local wellposedness Theorem 1.2 is an immediate consequence of Lemma below, which will be useful in the sequel.

**Lemma 1.3.** Let  $T > 0$  be a fixed finite time. For any  $t \in [0, T]$ , the velocity estimate is given by

$$(9) \quad \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} + \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} \leq C \|u_0\|_{B_{2,1}^{\frac{1}{2}}} + \int_0^t (\|\nabla u(t')\|_{L^\infty} + W(t')) \|u(t')\|_{B_{2,1}^{\frac{1}{2}}} dt',$$

where

$$W(t') \stackrel{\text{def}}{=} 2^{2m} \|a\|_{L_{t'}^\infty(L^\infty)}^2 + 2^{8m} \|a\|_{L_{t'}^\infty(L^2)}^4 (1 + \|u\|_{L_{t'}^\infty(B_{2,1}^{\frac{1}{2}})}^4 + \|a\|_{L_{t'}^\infty(L^\infty)}^4).$$

Two above results will provide us the local and uniqueness existence of a solution  $(a, u)$ . Concerning the global aspect to this solution, we shall use an energy method, which can be achieved by vertue of Theorem 1.4 below.

**Theorem 1.4** (Global wellposedness Theorem). Given the initial data  $(\rho_0, u_0)$  and two positive constants  $m$  and  $M$  such that

$$(10) \quad u_0 \in H^2(\mathbb{T}^3), \quad 0 < m \leq \rho_0(x) \leq M, \quad \text{and} \quad \int_{\mathbb{T}^3} \rho_0 u_0 = 0.$$

There exists a constant  $\varepsilon_0 > 0$  (depending on  $m$  and  $M$ ) such that if  $u_0$  satisfies the smallness condition  $\|u_0\|_{H^2} \leq \varepsilon_0$  then, the system (3) has a (unique) global solution  $(\rho, u)$  which satisfies for any  $(t, x) \in [0, +\infty[ \times \mathbb{T}^3$

$$(11) \quad \begin{aligned} 0 &< m \leq \rho(t, x) \leq M, \\ B_0(t) &\leq \|\sqrt{\rho_0} u_0\|_{L^2}^2, \\ B_1(t) &\leq C \|\nabla u_0\|_{L^2}^2, \\ B_2(t) &\leq C \left(1 + \|u_0\|_{H^2}^4\right) \|u_0\|_{H^2}^2 \exp\left(\|u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2\right) \end{aligned}$$

where  $B_0(t)$ ,  $B_1(t)$  and  $B_2(t)$  are defined by

$$(12) \quad B_0(T) \stackrel{\text{def}}{=} \sup_{t \in [0, T]} \|\sqrt{\rho} u(t)\|_{L^2}^2 + \int_0^T \int_{\mathbb{T}^3} |\nabla u(t, x)|^2 dx dt.$$

$$(13) \quad B_1(T) \stackrel{\text{def}}{=} \sup_{t \in [0, T]} \|\nabla u(t)\|_{L^2}^2 + \int_0^T \left( \|\sqrt{\rho} \partial_t u(t)\|_{L^2}^2 + \|\nabla \Pi(t)\|_{L^2}^2 \right) dt + \frac{1}{8} \int_0^T \|\nabla^2 u(t)\|_{L^2}^2 dt,$$

$$(14) \quad B_2(T) \stackrel{\text{def}}{=} \sup_{t \in [0, T]} \left( \frac{1}{2} \|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla \Pi(t)\|_{L^2}^2 + \frac{m}{3} \|\partial_t u(t)\|_{L^2}^2 \right) \\ + \frac{1}{4} \int_0^T \|\nabla \partial_t u(t)\|_{L^2}^2 dt + \frac{1}{2} \int_0^T \|\nabla^2 u(t)\|_{L^6}^2 dt + \int_0^T \|\nabla^2 \Pi(t)\|_{L^6}^2 dt.$$

**Remark 1.5.** *We shall prove the existence and global part by an energy method. We underline the very weak assumption (bounded from above and below) on the density we need. We refer the reader to [17] for the uniqueness proof.*

*Guideline of the proof and organisation of the paper.*

Firstly, we prove the local existence and uniqueness of a solution, under hypothesis of Theorem 1.2. Then, we underline that, provided  $\|u_0\|_{B_{2,1}^{\frac{1}{2}}}$  is small enough, the lifespan  $T^*(u_0)$  of the local solution associated with this data should be greater than 1. This is due to scaling argument. In addition, velocity estimate (8) implies

$$(15) \quad \exists t_1 \in [0, 1[ \quad \text{such that} \quad u(t_1) \in H^2 \quad \text{and} \quad \|u(t_1)\|_{H^2} \leq C \|u_0\|_{B_{2,1}^{\frac{1}{2}}}.$$

This stems from an interpolation argument, provided  $T^*(u_0) > 1$ . Indeed, assume we have proved there exists an unique solution  $u$  such that

$$u \in L^\infty([0, T], B_{2,1}^{\frac{1}{2}}) \cap L^1([0, T], B_{2,1}^{\frac{5}{2}}),$$

and thus,  $u$  belongs to  $L^{\frac{4}{3}}([0, T], H^2)$ , which provide the existence of the small time  $t_1$ , such that (15) is satisfied.

From this point, the strategy to deal with the global property of our system takes another direction than the strategy setting up in [4]. Indeed, we shall prove that, considering  $u(t_1)$  as an initial data in  $H^2$ , which is small enough (since  $\|u_0\|_{B_{2,1}^{\frac{1}{2}}}$  is supposed to be so) and thanks to Theorem 1.4 below,

there exists a global solution (the uniqueness is non necessary for what we need in the sequel).

Then, it remains to be seen that such a solution has the relevant regularity, namely the regularity demanding by Theorem 1.2. In others words, it is crucial to prove the propagation of the regularity of the density function  $a$ , from which we infer the regularity of the velocity, thanks to Lemma 1.3. To sum up, we will prove the existence of a global solution with the relevant regularity : this proves the uniqueness of such a solution.

The paper is structured as follows. In Section 2, we collect some basic facts on Littlewood Paley theory, Besov spaces and we will give the classical inequalities (well-known in the whole space  $\mathbb{R}^3$ ), in the case of the torus  $\mathbb{T}^3$ . In addition, we will stress on the important role of the average  $\bar{u}$ .

Section 3 is devoted to the proof of the main Theorem 1.1. Section 4 deals with the local wellposedness issue of the main theorem : we will prove Theorem 1.2. Section 5 provides the global wellposedness aspect of the main theorem, which will stem from the proof of Theorem 1.4. Let us mention we will only give in both two cases the a priori estimates. It means we skip the standard procedure of Friedrich's regularization. The point is that we deal with uniform estimates, in which we use a standard compactness argument.

## 2. TOOL BOX CONCERNING ESTIMATES ON THE TORUS $\mathbb{T}^3$

**Proposition 2.1.** *(Poincaré-Wirtinger inequality)*

*Let  $u$  be in  $H^1(\mathbb{T}^3)$  and mean free. Then we have :*

$$\|u\|_{L^2(\mathbb{T}^3)} \leq \|\nabla u\|_{L^2(\mathbb{T}^3)}.$$

*In particular, the  $\dot{H}^1(\mathbb{T}^3)$  and  $H^1(\mathbb{T}^3)$ -norms are equivalent, when  $\bar{u}$  is mean free.*

An obvious consequence of the Poincaré-Wirtinger inequality is the corollary below.

**Corollary 2.2.** *Let  $u$  be in  $H^1(\mathbb{T}^3)$ . Then we have :*

$$\|u - \bar{u}\|_{L^2(\mathbb{T}^3)} \leq \|\nabla u\|_{L^2(\mathbb{T}^3)}.$$

**Proposition 2.3.** *(Gagliardo-Nirenberg inequality)*

*In the whole space  $\mathbb{R}^3$  :  $\|u\|_{L^p} \leq \|u\|_{L^2}^{\frac{3}{p}-\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{3}{2}-\frac{3}{p}}$ , with  $2 \leq p \leq 6$ .*

*On the torus  $\mathbb{T}^3$  :  $\|u - \bar{u}\|_{L^p} \leq \|u\|_{L^2}^{\frac{3}{p}-\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{3}{2}-\frac{3}{p}}$ , with  $2 \leq p \leq 6$ .*

In particular, for  $p = 6$ , we find the Sobolev embeddings on the torus :

$$\|u - \bar{u}\|_{L^6(\mathbb{T}^3)} \leq C \|\nabla u\|_{L^2(\mathbb{T}^3)} \quad \text{instead of} \quad \|u\|_{L^6(\mathbb{R}^3)} \leq C \|\nabla u\|_{L^2(\mathbb{R}^3)}.$$

The following Lemma is fundamental in this paper. It highlights the crucial role playing by the average of the velocity. Because the framework of our work is the torus, we will need several times in the next, to have an estimate on the average. Actually, it provides a general method to compute the average of a quantity we are interesting in. We will call it *the average method* in the sequel.

**Lemma 2.4.** *Assuming that  $|\mathbb{T}^3| = 1$  and  $\int_{\mathbb{T}^3} \rho_0 u_0 = 0$ , we have :*

$$|\bar{u}(t)| \leq \frac{\|\rho_0 - \bar{\rho}_0\|_{L^2}}{|\bar{\rho}_0|} \|\nabla u(t)\|_{L^2}.$$

*Proof.* Let us consider the integral below and developp it

$$\int_{\mathbb{T}^3} (\rho - \bar{\rho})(t, x)(u - \bar{u})(t, x) dx = \int_{\mathbb{T}^3} \rho(t, x) u(t, x) - 2\bar{\rho}(t) \bar{u}(t) + \bar{\rho}(t) \bar{u}(t).$$

Thanks to (1.1) and (1.2), we have

$$\begin{aligned} \bar{u}(t) &= -\frac{1}{\bar{\rho}(t)} \int_{\mathbb{T}^3} (\rho - \bar{\rho})(t, x)(u - \bar{u})(t, x) dx \\ (16) \quad &= -\frac{1}{\bar{\rho}_0} \int_{\mathbb{T}^3} (\rho - \bar{\rho})(t, x)(u - \bar{u})(t) \\ &|\bar{u}(t)| \leq \frac{1}{|\bar{\rho}_0|} \|(\rho - \bar{\rho})(t)\|_{L^2} \|(u - \bar{u})(t)\|_{L^2}. \end{aligned}$$

Applying (1.3), we have

$$(17) \quad |\bar{u}(t)| \leq \frac{1}{|\bar{\rho}_0|} \|\rho_0 - \bar{\rho}_0\|_{L^2} \|(u - \bar{u})(t)\|_{L^2}.$$

Thanks to Poincaré-Wirtinger, we get :

$$(18) \quad |\bar{u}(t)| \leq \frac{\|\rho_0 - \bar{\rho}_0\|_{L^2}}{|\bar{\rho}_0|} \|\nabla u(t)\|_{L^2}.$$

□

**Proposition 2.5.** *Assuming that  $|\mathbb{T}^3| = 1$  and  $\int_{\mathbb{T}^3} \rho_0 u_0 = 0$ , therefore  $\|u(t)\|_{L^6} \leq C(\rho_0) \|\nabla u(t)\|_{L^2}$ .*

*Proof.*

$$\begin{aligned} (19) \quad \|u(t)\|_{L^6}^2 &\leq \|(u - \bar{u})(t)\|_{L^6}^2 + |\bar{u}(t)|^2 \\ &\leq C \|\nabla u(t)\|_{L^2}^2 + \frac{\|\rho_0 - \bar{\rho}_0\|_{L^2}^2}{\bar{\rho}_0^2} \|\nabla u(t)\|_{L^2}^2 \\ &\leq C(\rho_0) \|\nabla u(t)\|_{L^2}^2. \end{aligned}$$

□

**Proposition 2.6.** *If  $|\mathbb{T}^3| = 1$  and  $\int_{\mathbb{T}^3} \rho_0 u_0 = 0$ , then  $\|u(t)\|_{L^3} \leq C(\rho_0) \|\nabla u(t)\|_{L^2}$ .*

*Proof.* Arguments are similar as before. We introduce the average of  $u$  and we apply successively Gagliardo-Nirenberg and Poincaré-Wirtinger inequalities

$$\begin{aligned}
 \|u(t)\|_{L^3} &\leq \|(u - \bar{u})(t)\|_{L^3} + |\bar{u}(t)| \\
 &\leq \|(u - \bar{u})(t)\|_{L^2}^{\frac{1}{2}} \|\nabla(u - \bar{u})(t)\|_{L^2}^{\frac{1}{2}} + |\bar{u}(t)| \\
 &\leq \|\nabla u(t)\|_{L^2}^{\frac{1}{2}} \|\nabla u(t)\|_{L^2}^{\frac{1}{2}} + |\bar{u}(t)| \\
 &\leq \|\nabla u(t)\|_{L^2} + |\bar{u}(t)|.
 \end{aligned}
 \tag{20}$$

Concerning the term  $|\bar{u}(t)|$ , same computations as in Lemma 2.4 yield

$$\begin{aligned}
 \|u(t)\|_{L^3} &\leq \|\nabla u(t)\|_{L^2} + \frac{1}{|\bar{\rho}_0|} \|\rho_0 - \bar{\rho}_0\|_{L^2} \|(u - \bar{u})(t)\|_{L^2} \\
 &\leq C(\rho_0) \|\nabla u(t)\|_{L^2}.
 \end{aligned}
 \tag{21}$$

□

### 3. PROOF OF THE MAIN THEOREM

Assuming we have proved Theorems 1.2 and 1.4, we can prove the main Theorem. Firstly, notice that Theorem 1.2 implies

$$\exists t_1 \in [0, T], \quad u(t_1) \in H^2 \cap B_{2,1}^{\frac{1}{2}}, \quad \text{and} \quad \|u(t_1)\|_{H^2} \leq \|u_0\|_{B_{2,1}^{\frac{1}{2}}}.
 \tag{22}$$

Moreover, we have a fundamental information on  $a(t_1)$  :

$$a(t_1) \in B_{2,1}^{\frac{3}{2}} \cap L^\infty.
 \tag{23}$$

Let us underline that we have, by virtue of Remark 1.2,

$$\int_{\mathbb{T}^3} \frac{1}{1 + a(t_1)} u(t_1) = \int_{\mathbb{T}^3} \frac{1}{1 + a_0} u_0 = 0.
 \tag{24}$$

As a consequence, Theorem 1.4 implies there exists a global solution  $(\rho, w)$  of the system (2) associated with data

$$(\rho, w)_{t=0} \stackrel{\text{def}}{=} \left( \frac{1}{1 + a(t_1)}, u(t_1) \right).$$

First of all, we adopt the classical point of view : from the solution  $(\rho, w)$  of the system (2), we define the solution  $(a_w, w)$  of the system (4), given by

$$\rho \stackrel{\text{def}}{=} \frac{1}{1 + a_w}.$$

Therefore, it follows that the solution  $(a_w, w)$  is associated with the data  $(a(t_1), u(t_1))$ , which belongs to  $B_{2,1}^{\frac{3}{2}} \cap L^\infty \times H^2$ .

The goal is to prove the uniqueness of such a solution, which will come from the following regularity

$$\forall T \geq 0, \quad (a_w, w) \in \mathcal{C}([0, T], B_{2,1}^{\frac{3}{2}}) \times \mathcal{C}([0, T], B_{2,1}^{\frac{1}{2}}) \cap L^1([0, T], B_{2,1}^{\frac{5}{2}}).$$

Proving such a regularity on the density function and the velocity field provides us the uniqueness by virtue of local wellposedness Theorem 1.2. The point is the propagation of the regularity of  $a_w$ .

### 3.1. Propagation of the regularity of the density.

**Proposition 3.1.** *Let  $T > 0$  be a time fixed. Then,  $\forall t \in [0, T]$ ,  $a_w(t) \in B_{2,1}^{\frac{3}{2}}$ .*

*Proof.* Applying the frequencies localization operator  $\Delta_q$  on the transport equation, we get

$$(25) \quad \partial_t \Delta_q a_w + w \cdot \nabla \Delta_q a_w = -[\Delta_q, w \cdot \nabla] a_w.$$

Taking the  $L^2$ -inner product with  $\Delta_q a$ , the divergence-free condition implies that

$$(26) \quad \frac{1}{2} \frac{d}{dt} \|\Delta_q a_w\|_{L^2}^2 \leq \|\Delta_q a_w\|_{L^2} \|[\Delta_q, w \cdot \nabla] a_w\|_{L^2}.$$

By virtue of Gronwall's Lemma 6.1 (given in the appendix), we infer that

$$(27) \quad 2^{\frac{3q}{2}} \|\Delta_q a_w(t)\|_{L^2} \leq 2^{\frac{3q}{2}} \|\Delta_q a(t_1)\|_{L^2} + 2^{\frac{3q}{2}} \int_0^t \|[\Delta_q, w \cdot \nabla] a_w\|_{L^2} dt'.$$

Therefore, by some classical estimate of the commutator (see Lemma 2.100 in [5]), we get

$$(28) \quad \|a_w(t)\|_{B_{2,1}^{\frac{3}{2}}} \leq \|a(t_1)\|_{B_{2,1}^{\frac{3}{2}}} + C \int_0^t (\|a_w(t')\|_{B_{2,1}^{\frac{3}{2}}} \|\nabla w(t')\|_{L^\infty} + \|\nabla a_w(t')\|_{L^3} \|\nabla w(t')\|_{B_{6,1}^{\frac{1}{2}}}) dt'.$$

From the following embedding  $B_{2,1}^{\frac{3}{2}} \hookrightarrow B_{3,1}^1$  which holds in dimension 3, Gronwall Lemma yields

$$(29) \quad \|a_w(t)\|_{B_{2,1}^{\frac{3}{2}}} \leq \|a(t_1)\|_{B_{2,1}^{\frac{3}{2}}} \exp \left( C \int_0^t (\|\nabla w(t')\|_{L^\infty} + \|\nabla w(t')\|_{B_{6,1}^{\frac{1}{2}}}) dt' \right).$$

It remains to be checked that  $\int_0^t \|\nabla w(t')\|_{L^\infty} dt'$  and  $\int_0^t \|\nabla w(t')\|_{B_{6,1}^{\frac{1}{2}}} dt'$  exist for any time. This stems from energy method applying on  $w$ , thanks to Theorem 1.4. Concerning the term  $\int_0^t \|\nabla w(t')\|_{L^\infty} dt'$ , an interpolation argument gives rise to

$$(30) \quad \begin{aligned} \int_0^t \|\nabla w(t')\|_{L^\infty} dt' &\leq \int_0^t \|\nabla w(t')\|_{L^2}^{\frac{1}{4}} \|\nabla^2 w(t')\|_{L^6}^{\frac{3}{4}} dt' \\ &\leq \frac{1}{4} \int_0^t \|\nabla w(t')\|_{L^2} dt' + \frac{3}{4} \int_0^t \|\nabla^2 w(t')\|_{L^6} dt', \end{aligned}$$

and thanks to Hölder's inequality, we get

$$(31) \quad \int_0^t \|\nabla w(t')\|_{L^\infty} \leq C t^{\frac{1}{2}} (\|\nabla w(t')\|_{L_t^2(L^2)} + \|\nabla^2 w(t')\|_{L_t^2(L^6)}).$$

By virtue of Theorem 1.4,  $\|\nabla w\|_{L_t^2(L^2)} \leq C \|u(t_1)\|_{L^2}$  and  $\|\nabla^2 w\|_{L_t^2(L^6)} \leq C \|u(t_1)\|_{H^2}$ , therefore,

$$(32) \quad \int_0^t \|\nabla w(t')\|_{L^\infty} dt' \leq C t^{\frac{1}{2}} \|u(t_1)\|_{H^2}.$$

Concerning the term  $\int_0^t \|\nabla w(t')\|_{B_{6,1}^{\frac{1}{2}}} dt'$ , arguments are similar to the others ones and lead us to

$$(33) \quad \begin{aligned} \int_0^t \|\nabla w(t')\|_{B_{6,1}^{\frac{1}{2}}} dt' &\leq \int_0^t \|w(t')\|_{B_{6,\infty}^{-1}}^{\frac{1}{2}} \|w(t')\|_{B_{6,\infty}^2}^{\frac{1}{2}} dt' \\ &\leq \frac{1}{2} \int_0^t \|w(t')\|_{B_{6,\infty}^{-1}} dt' + \frac{1}{2} \int_0^t \|w(t')\|_{B_{6,\infty}^2} dt'. \end{aligned}$$

Notice we have the following embeddings

$$(34) \quad L^2 \hookrightarrow B_{6,\infty}^{-1} \quad \text{and} \quad L^6 \hookrightarrow B_{6,\infty}^0,$$



from which we infer that (thanks to Theorem 1.4)

$$\begin{aligned}
 \int_0^t \|\nabla w\|_{B_{6,1}^{\frac{1}{2}}} &\leq \frac{1}{2} \int_0^t \|w\|_{L^2} + \frac{1}{2} \int_0^t \|\nabla^2 w\|_{L^6} \\
 (35) \quad &\leq \frac{t}{2} \|w\|_{L_t^\infty(L^2)} + \frac{1}{2} t^{\frac{1}{2}} \|\nabla^2 w\|_{L_t^2(L^6)} \\
 &\leq \frac{1}{2} t \|u(t_1)\|_{L^2} + \frac{1}{2} t^{\frac{1}{2}} \|u(t_1)\|_{H^2}.
 \end{aligned}$$

Choosing  $t$  small enough such that  $t \leq t^{\frac{1}{2}}$ , we get

$$\int_0^t \|\nabla w(t')\|_{B_{6,1}^{\frac{1}{2}}} dt' \leq C t^{\frac{1}{2}} \|u(t_1)\|_{H^2}.$$

This yields to the desired estimate

$$(36) \quad \|a_w(t)\|_{B_{2,1}^{\frac{3}{2}}} \leq \|a(t_1)\|_{B_{2,1}^{\frac{3}{2}}} \exp(C t^{\frac{1}{2}} \|u(t_1)\|_{H^2}).$$

This concludes the proof on the propagation of the regularity on the density function.

**3.2. Regularity of the velocity field.** Holding the regularity on the density, we are allowed to apply Lemma 1.3, which gives rise to the following estimate, available, for any  $t \in [0, T]$ , where  $T$  is a fixed finite time.

$$(37) \quad \|w\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + \|w\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} + \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} \leq C \|u(t_1)\|_{B_{2,1}^{\frac{1}{2}}} + C \int_0^t (\|\nabla w(t')\|_{L^\infty} + W(t')) \|w(t')\|_{B_{2,1}^{\frac{1}{2}}} dt',$$

where

$$W(t') \stackrel{\text{def}}{=} 2^{2m} \|a_w\|_{L_{t'}^\infty(L^\infty)}^2 + 2^{8m} \|a_w\|_{L_{t'}^\infty(L^2)}^4 (1 + \|w\|_{L_{t'}^\infty(B_{2,1}^{\frac{1}{2}})}^4 + \|a_w\|_{L_{t'}^\infty(L^\infty)}^4).$$

We deduce from this estimate, by Gronwall Lemma,

$$(38) \quad \|w\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + \|w\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} + \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} \leq C \|u(t_1)\|_{B_{2,1}^{\frac{1}{2}}} \exp(\|\nabla w\|_{L_t^1(L^\infty)} + t W(t)).$$

Concerning the term  $W(t)$ , on the one hand, by the transport equation, we get immediately

$$\|a_w(t, \cdot)\|_{L^2} = \|a_w(0, \cdot)\|_{L^2},$$

which is bounded by  $\|a_w(0)\|_{B_{2,1}^{\frac{3}{2}}}$ , since spaces are inhomogeneous. On the other hand, by an interpolation argument, one has

$$\begin{aligned}
 \|w\|_{B_{2,1}^{\frac{1}{2}}} &\leq \|w\|_{B_{2,\infty}^0}^{\frac{1}{2}} \|w\|_{B_{2,\infty}^1}^{\frac{1}{2}} \\
 &\leq \|w\|_{L^2}^{\frac{1}{2}} \|w\|_{H^1}^{\frac{1}{2}}.
 \end{aligned}$$

It follows that, by virtue of Theorem 1.4,

$$\begin{aligned}
 (39) \quad \|w\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} &\leq \|w\|_{L_t^\infty(L^2)}^{\frac{1}{2}} \|w\|_{L_t^\infty(H^1)}^{\frac{1}{2}} \\
 &\leq \|u(t_1)\|_{L^2}^{\frac{1}{2}} (\|u(t_1)\|_{L^2}^{\frac{1}{2}} + \|\nabla u(t_1)\|_{L^2}^{\frac{1}{2}}).
 \end{aligned}$$

It results from these simple computations that the factor  $W(t)$  is bounded by

$$\forall t \in [0, T], \quad W(t) \leq C \|u(t_1)\|_{H^2}.$$

As it has been already noticed, the term  $\|\nabla w\|_{L_t^1(L^\infty)}$  satisfies

$$(40) \quad \int_0^t \|\nabla w(t')\|_{L^\infty} dt' \leq C t^{\frac{1}{2}} \|u(t_1)\|_{H^2}.$$

It results from all of this, that for any  $t \in [0, T]$ , we have

$$(41) \quad \|w\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + \|w\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} + \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} \leq C \|u(t_1)\|_{B_{2,1}^{\frac{1}{2}}} \exp(C t^{\frac{1}{2}} \|u(t_1)\|_{H^2}).$$

Combining with the estimate on the density function (36), we get  $t \in [0, T]$ , for a fixed time  $T > 0$

$$(42) \quad \|a_w\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})} + \|w\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + \|w\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} + \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} \leq C \|u(t_1)\|_{B_{2,1}^{\frac{1}{2}}} \exp(C t^{\frac{1}{2}} \|u(t_1)\|_{H^2}).$$

This ends up the proof of Theorem 1.1. □

#### 4. PROOF OF THE LOCAL WELLPOSEDNESS PART OF THE MAIN THEOREM

This section is devoted to the proof of Theorem 1.2. We give only the proof of the existence part of the theorem, since the uniqueness part has been already proved in [3]. We only mention the start point of the uniqueness proof.

**4.1. Existence part.** The existence proof can be achieved by a regularization process (e.g. Friedrich method). The idea is classical : we build smooth approximate solutions, perform uniform estimates on them. A compactness argument leads us to the proof of the existence of a solution of 4. We skip this part and provide some a priori estimates for smooth enough solution  $(a, u)$ .

Let us start by proving the estimate (7) on the density. Applying the frequencies localization operator  $\Delta_q$  on the transport equation, we get

$$\partial_t \Delta_q a + u \cdot \nabla \Delta_q a = -[\Delta_q, u \cdot \nabla] a.$$

Taking the  $L^2$ -inner product with  $\Delta_q a$ , the divergence-free condition implies that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta_q a\|_{L^2}^2 &= -([\Delta_q, u \cdot \nabla] a | \Delta_q a)_{L^2} \\ &\leq \|\Delta_q a\|_{L^2} \|[\Delta_q, u \cdot \nabla] a\|_{L^2}. \end{aligned}$$

By virtue of Gronwall's Lemma 6.1 (given in the appendix), we infer that

$$2^{\frac{3q}{2}} \|\Delta_q a\|_{L^2} \leq 2^{\frac{3q}{2}} \|\Delta_q a_0\|_{L^2} + 2^{\frac{3q}{2}} \int_0^t \|[\Delta_q, u \cdot \nabla] a\|_{L^2} dt'.$$

A classical commutator estimate (see for instance Lemma 2.100 in [5]) shows there exists a sequence  $(c_q)$  belonging to  $\ell^1(\mathbb{Z})$  such that

$$2^{\frac{3q}{2}} \|[\Delta_q, u \cdot \nabla] a\|_{L^2} \leq c_q \|a\|_{B_{2,1}^{\frac{3}{2}}} \|u\|_{B_{2,1}^{\frac{5}{2}}},$$

and therefore,

$$2^{\frac{3q}{2}} \int_0^t \|[\Delta_q, u \cdot \nabla] a\|_{L^2} dt' \leq \sup_t c_q(t) \int_0^t \|a(t')\|_{B_{2,1}^{\frac{3}{2}}} \|u(t')\|_{B_{2,1}^{\frac{5}{2}}} dt'.$$

By summing on  $q \in \mathbb{Z}$ , we get

$$\|a\|_{B_{2,1}^{\frac{3}{2}}} \leq \|a_0\|_{B_{2,1}^{\frac{3}{2}}} + C \int_0^t \|a(t')\|_{B_{2,1}^{\frac{3}{2}}} \|u(t')\|_{B_{2,1}^{\frac{5}{2}}} dt'.$$

The classical Gronwall's Lemma yields the proof of (7).

Let us prove estimate (8) on the velocity. Actually, we prove Lemma 1.3, which is a bit more general than we want to get.

*Proof of Lemma 1.3.*

We may rewrite the system (4), after decomposing  $(1 + a)$  into  $(1 + S_m a) + (a - S_m a)$ .

$$(43) \quad \partial_t u + u \cdot \nabla u - (1 + S_m a) \Delta u + (1 + S_m a) \nabla \Pi = (a - S_m a)(\Delta u - \nabla \Pi)$$

Notice that  $(1 + S_m a) \nabla \Pi = \nabla((1 + S_m a) \Pi) - \Pi \nabla S_m a$ , which implies

$$\partial_t u + u \cdot \nabla u - (1 + S_m a) \Delta u + \nabla((1 + S_m a) \Pi) = (a - S_m a)(\Delta u - \nabla \Pi) + \Pi \nabla S_m a.$$

Let us introduce the notation  $E_m \stackrel{\text{def}}{=} (a - S_m a)(\Delta u - \nabla \Pi)$ . We reduce the problem to the system below

$$(44) \quad \begin{cases} \partial_t u + u \cdot \nabla u - (1 + S_m a) \Delta u + \nabla((1 + S_m a) \Pi) &= E_m + \Pi \nabla S_m a. \\ \operatorname{div} u &= 0 \\ (a, u)|_{t=0} &= (a_0, u_0), \end{cases}$$

*Step 1: Frequency localization.*

Applying the operator  $\Delta_q$  in (44), we localize the velocity in a ring, with a size  $2^q$ , and we get

$$\partial_t \Delta_q u + \Delta_q(u \cdot \nabla u) - \Delta_q((1 + S_m a) \Delta u) + \Delta_q(\nabla((1 + S_m a) \Pi)) = \Delta_q E_m + \Delta_q(\Pi \nabla S_m a).$$

By definition of the commutator  $\Delta_q(u \cdot \nabla u) \stackrel{\text{def}}{=} u \cdot \nabla \Delta_q u + [\Delta_q, u \cdot \nabla] u$ , this gives

$$\begin{aligned} \partial_t \Delta_q u + u \cdot \nabla \Delta_q u - \Delta_q((1 + S_m a) \Delta u) + \Delta_q(\nabla((1 + S_m a) \Pi)) &= -[\Delta_q, u \cdot \nabla] u + \Delta_q E_m \\ &\quad + \Delta_q(\Pi \nabla S_m a). \end{aligned}$$

In particular, a simple computation gives

$$-\Delta_q((1 + S_m a) \Delta u) = -\operatorname{div}((1 + S_m a) \Delta_q \nabla u) - \operatorname{div}([\Delta_q, S_m a] \nabla u) + \Delta_q(\nabla S_m a \nabla u).$$

As a consequence, we get

$$(45) \quad \begin{aligned} \partial_t \Delta_q u + u \cdot \nabla \Delta_q u - \operatorname{div}((1 + S_m a) \Delta_q \nabla u) + \Delta_q(\nabla((1 + S_m a) \Pi)) &= -[\Delta_q, u \cdot \nabla] u + \Delta_q E_m \\ &\quad + \Delta_q(\Pi \nabla S_m a) + \operatorname{div}([\Delta_q, S_m a] \nabla u) - \Delta_q(\nabla S_m a \nabla u). \end{aligned}$$

Let us take the  $L^2$  inner product with  $\Delta_q u$  in the above equation (45). Because of the divergence free condition, we have

$$(u \cdot \nabla \Delta_q u | \Delta_q u)_{L^2} = 0 \quad \text{and} \quad (\Delta_q(\nabla((1 + S_m a) \Pi)) | \Delta_q u)_{L^2} = 0.$$

As a result,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta_q u\|_{L^2}^2 + \int_{\mathbb{T}^3} (1 + S_m a) |\Delta_q \nabla u|^2 dx &\leq \|\Delta_q u\|_{L^2} \left( \|[\Delta_q, u \cdot \nabla] u\|_{L^2} + \|\Delta_q E_m\|_{L^2} + \|\Delta_q(\Pi \nabla S_m a)\|_{L^2} \right. \\ &\quad \left. + 2^q \|[\Delta_q, S_m a] \nabla u\|_{L^2} + \|\Delta_q(\nabla S_m a \nabla u)\|_{L^2} \right) \end{aligned}$$

Let us point that  $1 + S_m a = 1 + a + S_m a - a$ . As we assume that  $S_m a - a$  is small enough in norm  $L_t^\infty(B_{2,1}^{\frac{3}{2}})$ , it follows that

$$1 + S_m a \geq \frac{b}{2},$$

which along with Lemma 6.3, ensures that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta_q u\|_{L^2}^2 + \frac{b}{2} 2^{2q} \|\Delta_q u\|_{L^2}^2 &\leq \|\Delta_q u\|_{L^2} \left( \|[\Delta_q, u \cdot \nabla] u\|_{L^2} + \|\Delta_q E_m\|_{L^2} + \|\Delta_q(\Pi \nabla S_m a)\|_{L^2} \right. \\ &\quad \left. + 2^q \|[\Delta_q, S_m a] \nabla u\|_{L^2} + \|\Delta_q(\nabla S_m a \nabla u)\|_{L^2} \right). \end{aligned}$$

Applying a Gronwall's argument, we get

$$\begin{aligned} \frac{d}{dt} \|\Delta_q u\|_{L^2} + \frac{b}{2} 2^{2q} \|\Delta_q u\|_{L^2} &\leq \|[\Delta_q, u \cdot \nabla] u\|_{L^2} + \|\Delta_q E_m\|_{L^2} + \|\Delta_q (\Pi \nabla S_m a)\|_{L^2} \\ &\quad + 2^q \|[\Delta_q, S_m a] \nabla u\|_{L^2} + \|\Delta_q (\nabla S_m a \nabla u)\|_{L^2}. \end{aligned}$$

An integration in time yields

$$\begin{aligned} 2^{\frac{q}{2}} \|\Delta_q u\|_{L^2} + C b 2^{\frac{5q}{2}} \int_0^t \|\Delta_q u\|_{L^2} dt' &\leq 2^{\frac{q}{2}} \|\Delta_q u_0\|_{L^2} + \int_0^t 2^{\frac{q}{2}} \|[\Delta_q, u \cdot \nabla] u\|_{L^2} dt' \\ &\quad + \int_0^t 2^{\frac{q}{2}} \|\Delta_q E_m\|_{L^2} dt' + \int_0^t 2^{\frac{q}{2}} \|\Delta_q (\Pi \nabla S_m a)\|_{L^2} dt' \\ &\quad + \int_0^t 2^{\frac{3q}{2}} \|[\Delta_q, S_m a] \nabla u\|_{L^2} dt' + \int_0^t 2^{\frac{q}{2}} \|\Delta_q (\nabla S_m a \nabla u)\|_{L^2} dt'. \end{aligned}$$

Taking the supremum in time and then summing on  $q \in \mathbb{Z}$  provides us the norm  $\|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})}$  and thus

$$\begin{aligned} (46) \quad \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + C b \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} &\leq \|u_0\|_{B_{2,1}^{\frac{1}{2}}} + \|E_m\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} + \|\Pi \nabla S_m a\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} \\ &\quad + \sum_{q \in \mathbb{Z}} 2^{\frac{q}{2}} \|[\Delta_q, u \cdot \nabla] u\|_{L_t^1(L^2)} \\ &\quad + \sum_{q \in \mathbb{Z}} 2^{\frac{3q}{2}} \|[\Delta_q, S_m a] \nabla u\|_{L_t^1(L^2)} + \|\nabla S_m a \nabla u\|_{L_t^1(B_{2,1}^{\frac{1}{2}})}. \end{aligned}$$

*Step 2: Estimate of each term in the right-hand-side of the above inequality.*

★ Estimate of  $\|E_m\|_{L_t^1(B_{2,1}^{\frac{1}{2}})}$

Product laws in Besov spaces (cf Lemma 6.2 in Appendix) yield

$$\begin{aligned} (47) \quad \|E_m\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} &\leq C \|a - S_m a\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})} \|\Delta u - \nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} \\ &\leq C \|a - S_m a\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})} \left( \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} + \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} \right). \end{aligned}$$

★ Estimate of  $\|\Pi \nabla S_m a\|_{L_t^1(B_{2,1}^{\frac{1}{2}})}$

Concerning the pressure term, as it is defined up to a constant, we can assume it is mean free. Same remark holds for the term  $\|\nabla S_m a\|_{B_{2,2}^1}$ , since obviously the term  $\nabla S_m a$  is mean free. In this way, the norms  $\|\cdot\|_{B_{2,2}^1}$  and  $\|\cdot\|_{\dot{B}_{2,2}^1}$  are equivalent. By virtue of paradifferential calculus in inhomogeneous Besov norm, we get

$$\begin{aligned} (48) \quad \|\Pi \nabla S_m a\|_{B_{2,1}^{\frac{1}{2}}} &\leq C \|\Pi\|_{B_{2,2}^1} \|\nabla S_m a\|_{B_{2,2}^1} \\ &\leq C \|\Pi\|_{\dot{B}_{2,2}^1} \|\nabla S_m a\|_{\dot{B}_{2,2}^1} \\ &= C \|\nabla \Pi\|_{L^2} \|\nabla S_m a\|_{\dot{H}^1}, \end{aligned}$$

which leads to

$$\|\Pi \nabla S_m a\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} \leq C \|\nabla \Pi\|_{L_t^1(L^2)} \|\nabla S_m a\|_{L_t^\infty(\dot{H}^1)}.$$

★ Estimate of  $\|\nabla S_m a \nabla u\|_{L_t^1(B_{2,1}^{\frac{1}{2}})}$ . Above arguments still provide

$$\begin{aligned} (49) \quad \|\nabla S_m a \nabla u\|_{B_{2,1}^{\frac{1}{2}}} &\leq C \|\nabla S_m a\|_{B_{2,2}^1} \|\nabla u\|_{B_{2,2}^1} \\ &\leq C \|\nabla u\|_{\dot{H}^1} \|\nabla S_m a\|_{\dot{H}^1}. \end{aligned}$$

Therefore, we deduce that

$$\|\nabla S_m a \nabla u\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} \leq C \|u\|_{L_t^1(\dot{H}^2)} \|\nabla S_m a\|_{L_t^\infty(\dot{H}^1)}.$$

★ Estimate of  $\sum_{q \in \mathbb{Z}} 2^{\frac{q}{2}} \|\Delta_q, u \cdot \nabla\| u\|_{L_t^1(L^2)}$ . By virtue of commutator estimate, we infer that

$$\|\Delta_q, u \cdot \nabla\| u\|_{L^2} \leq C d_q 2^{-\frac{q}{2}} \|\nabla u\|_{B_{2,1}^{\frac{3}{2}}} \|u\|_{B_{2,1}^{\frac{1}{2}}}.$$

Therefore, we deduce that

$$\sum_{q \in \mathbb{Z}} 2^{\frac{q}{2}} \|\Delta_q, u \cdot \nabla\| u\|_{L_t^1(L^2)} \leq C \int_0^t \|\nabla u(t')\|_{L^\infty} \|u(t')\|_{B_{2,1}^{\frac{1}{2}}} dt'.$$

★ Estimate of  $\sum_{q \in \mathbb{Z}} 2^{\frac{3q}{2}} \|\Delta_q, S_m a\| \nabla u\|_{L_t^1(L^2)}$ . We can prove the estimate below (see Lemma 6.3)

$$(50) \quad \sum_{q \in \mathbb{Z}} 2^{\frac{3q}{2}} \|\Delta_q, S_m a\| \nabla u\|_{L_t^1(L^2)} \leq C 2^m \|a\|_{L_t^\infty(L^\infty)} \|u\|_{L_t^1(B_{2,1}^{\frac{3}{2}})} + 2^{2m} \|a\|_{L_t^\infty(L^2)} \|u\|_{L_t^1(\dot{H}^2)}.$$

Plugging all the above estimates in (46), we finally get

$$(51) \quad \begin{aligned} \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + C b \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} &\leq \|u_0\|_{B_{2,1}^{\frac{1}{2}}} + \|a - S_m a\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})} \left( \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} + \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} \right) \\ &\quad + \int_0^t \|\nabla u(t')\|_{L^\infty} \|u(t')\|_{B_{2,1}^{\frac{1}{2}}} dt' + 2^m \|a\|_{L_t^\infty(L^\infty)} \|u\|_{L_t^1(B_{2,1}^{\frac{3}{2}})} \\ &\quad + 2^{2m+1} \|a\|_{L_t^\infty(L^2)} \left( \|u\|_{L_t^1(\dot{H}^2)} + \|\nabla \Pi\|_{L_t^1(L^2)} \right), \end{aligned}$$

where we have used

$$\|\nabla S_m a\|_{L_t^\infty(\dot{H}^1)} = \|\nabla^2 S_m a\|_{L_t^\infty(L^2)} \leq 2^{2m} \|a\|_{L_t^\infty(L^2)}.$$

*Step 3: Estimate of  $\|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})}$ .*

We take the divergence operator in (43) and thus

$$\operatorname{div}((1 + S_m a) \nabla \Pi) = -\operatorname{div}(u \cdot \nabla u) + \Delta u \cdot \nabla S_m a + \operatorname{div}((S_m a - a)(\nabla \Pi - \Delta u)).$$

Applying the operator  $\Delta_q$  and taking the  $L^2$  inner product with  $\Delta_q \Pi$  yield

$$\begin{aligned} (\Delta_q((1 + S_m a) \nabla \Pi) | \Delta_q \Pi)_{L^2} &= (\Delta_q(u \cdot \nabla u) | \Delta_q \nabla \Pi)_{L^2} + (\Delta_q(\Delta u \cdot \nabla S_m a) | \Delta_q \Pi)_{L^2} \\ &\quad + (\Delta_q((S_m a - a) \nabla \Pi) | \Delta_q \nabla \Pi)_{L^2} - (\Delta_q((S_m a - a) \Delta u) | \Delta_q \nabla \Pi)_{L^2}. \end{aligned}$$

In particular, the left-hand-side can be rewritten and bounded from below as follows

$$\begin{aligned} (\Delta_q((1 + S_m a) \nabla \Pi) | \Delta_q \nabla \Pi)_{L^2} &= (\Delta_q \nabla \Pi | \Delta_q \nabla \Pi)_{L^2} + ([\Delta_q, S_m a] \nabla \Pi | \Delta_q \nabla \Pi)_{L^2} \\ &\quad + (S_m a \Delta_q \nabla \Pi | \Delta_q \nabla \Pi)_{L^2} \\ &= ((1 + S_m a) \Delta_q \nabla \Pi | \Delta_q \nabla \Pi)_{L^2} + ([\Delta_q, S_m a] \nabla \Pi | \Delta_q \nabla \Pi)_{L^2} \\ &\geq b \|\Delta_q \nabla \Pi\|_{L^2}^2 + ([\Delta_q, S_m a] \nabla \Pi | \Delta_q \nabla \Pi)_{L^2}. \end{aligned}$$

It follows

$$(52) \quad \begin{aligned} b \|\Delta_q \nabla \Pi\|_{L^2}^2 &\leq \|\Delta_q \nabla \Pi\|_{L^2} \left( \|\Delta_q(u \cdot \nabla u)\|_{L^2} + \|\Delta_q((S_m a - a) \nabla \Pi)\|_{L^2} \right. \\ &\quad \left. + \|\Delta_q((S_m a - a) \Delta u)\|_{L^2} + \|[\Delta_q, S_m a] \nabla \Pi\|_{L^2} \right) \\ &\quad + \|\Delta_q \Pi\|_{L^2} \|\Delta_q(\Delta u \cdot \nabla S_m a)\|_{L^2}. \end{aligned}$$

In particular, Lemma 6.3 provides the inequality below

$$\|\Delta_q \Pi\|_{L^2} \lesssim 2^{-q} \|\Delta_q \nabla \Pi\|_{L^2},$$

which gives rise to

$$(53) \quad \begin{aligned} b \|\Delta_q \nabla \Pi\|_{L^2} &\leq \|\Delta_q (u \cdot \nabla u)\|_{L^2} + \|\Delta_q ((S_m a - a) \nabla \Pi)\|_{L^2} + \|\Delta_q ((S_m a - a) \Delta u)\|_{L^2} \\ &\quad + \|\Delta_q, S_m a\| \nabla \Pi\|_{L^2} + 2^{-q} \|\Delta_q (\Delta u \cdot \nabla S_m a)\|_{L^2}. \end{aligned}$$

Multiplying by  $2^{\frac{q}{2}}$  and summing on  $q \in \mathbb{Z}$ , we have

$$\begin{aligned} b \|\nabla \Pi\|_{B_{2,1}^{\frac{3}{2}}} &\lesssim \|u \cdot \nabla u\|_{B_{2,1}^{\frac{1}{2}}} + \|(S_m a - a) \nabla \Pi\|_{B_{2,1}^{\frac{1}{2}}} + \|(S_m a - a) \Delta u\|_{B_{2,1}^{\frac{1}{2}}} \\ &\quad + \|\Delta u \cdot \nabla S_m a\|_{B_{2,1}^{-\frac{1}{2}}} + \sum_{q \in \mathbb{Z}} 2^{\frac{q}{2}} \|\Delta_q, S_m a\| \nabla \Pi\|_{L^2}. \end{aligned}$$

Notice that

$$\|\Delta u \cdot \nabla S_m a\|_{B_{2,1}^{-\frac{1}{2}}} \leq C \|\nabla S_m a\|_{\dot{H}^1} \|\Delta u\|_{L^2}.$$

On the one hand, product laws in Besov spaces (cf Lemma 6.2) give

$$\begin{aligned} \|u \cdot \nabla u\|_{B_{2,1}^{\frac{1}{2}}} &\leq \|u\|_{B_{2,1}^{\frac{1}{2}}} \|\nabla u\|_{L^\infty} . \\ \|(S_m a - a) \Delta u\|_{B_{2,1}^{\frac{1}{2}}} &\leq C \|(S_m a - a)\|_{B_{2,1}^{\frac{3}{2}}} \|\Delta u\|_{B_{2,1}^{\frac{1}{2}}} . \\ \|(S_m a - a) \nabla \Pi\|_{B_{2,1}^{\frac{1}{2}}} &\leq C \|(S_m a - a)\|_{B_{2,1}^{\frac{3}{2}}} \|\nabla \Pi\|_{B_{2,1}^{\frac{1}{2}}} . \end{aligned}$$

On the other hand, a classical commutator estimate yields

$$\sum_{q \in \mathbb{Z}} 2^{\frac{q}{2}} \|\Delta_q, S_m a\| \nabla \Pi\|_{L^2} \leq C \|\nabla S_m a\|_{\dot{H}^1} \|\nabla \Pi\|_{L^2}.$$

As a result, previous estimates imply

$$(54) \quad \begin{aligned} b \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} &\lesssim \int_0^t \|u(t')\|_{B_{2,1}^{\frac{1}{2}}} \|\nabla u(t')\|_{L^\infty} dt' + \|(S_m a - a)\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})} \|\Delta u\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} \\ &\quad + \|(S_m a - a)\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})} \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} + \|\nabla S_m a\|_{L_t^\infty(\dot{H}^1)} \left( \|\nabla \Pi\|_{L_t^1(L^2)} + \|\Delta u\|_{L_t^1(L^2)} \right). \end{aligned}$$

The smallness condition on  $\|(S_m a - a)\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})}$  allows to write

$$\begin{aligned} \frac{b}{2} \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} &\lesssim \int_0^t \|u(t')\|_{B_{2,1}^{\frac{1}{2}}} \|\nabla u(t')\|_{L^\infty} dt' + \|(S_m a - a)\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})} \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} \\ &\quad + \|\nabla S_m a\|_{L_t^\infty(\dot{H}^1)} \left( \|\nabla \Pi\|_{L_t^1(L^2)} + \|\Delta u\|_{L_t^1(L^2)} \right). \end{aligned}$$

Obviously,  $\|\nabla S_m a\|_{L_t^\infty(\dot{H}^1)} \leq C 2^{2m} \|a\|_{L_t^\infty(L^2)}$ . Therefore,

$$(55) \quad \begin{aligned} \frac{b}{2} \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} &\lesssim \int_0^t \|u(t')\|_{B_{2,1}^{\frac{1}{2}}} \|\nabla u(t')\|_{L^\infty} dt' + \|(S_m a - a)\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})} \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} \\ &\quad + 2^{2m} \|a\|_{L_t^\infty(L^2)} \left( \|\nabla \Pi\|_{L_t^1(L^2)} + \|u\|_{L_t^1(\dot{H}^2)} \right). \end{aligned}$$

This ends up the estimate on the pressure term in  $L_t^1(B_{2,1}^{\frac{1}{2}})$ -norm. It is left with estimate the pressure term in the  $L_t^1(L^2)$ -norm, in order to get rid of it in the above estimate, and thus, it is likely to applying with success Gronwall Lemma in the estimate of the velocity term.

*Step 4: Estimate of  $\|\nabla\Pi\|_{L_t^1(L^2)}$ .*

Once again, we take the divergence in the momentum equation, and the  $\dot{H}^{-1}$ -norm, so that we get

$$\|\operatorname{div}((1 + S_m a)\nabla\Pi)\|_{\dot{H}^{-1}} \leq \|\operatorname{div}(u \cdot \nabla u)\|_{\dot{H}^{-1}} + \|\Delta u \cdot \nabla S_m a\|_{\dot{H}^{-1}} + \|\operatorname{div}((S_m a - a)(\nabla\Pi - \Delta u))\|_{\dot{H}^{-1}}.$$

We recall that the smallness condition implies that  $(1 + S_m a) \geq \frac{b}{2}$  and thus

$$b \|\nabla\Pi\|_{L^2} \leq C \|(1 + S_m a)\nabla\Pi\|_{L^2} \leq C \|u \cdot \nabla u\|_{L^2} + \|\Delta u \cdot \nabla S_m a\|_{\dot{H}^{-1}} + \|(S_m a - a)(\nabla\Pi - \Delta u)\|_{L^2}.$$

Thanks to the smallness condition and product law, we have

$$(56) \quad \frac{b}{2} \|\nabla\Pi\|_{L^2} \lesssim \|u\|_{L^3} \|\nabla u\|_{L^6} + \|\Delta u \cdot \nabla S_m a\|_{\dot{H}^{-1}} + \|(S_m a - a)\Delta u\|_{L^2}.$$

On the one hand, Gagliardo-Nirenberg inequality (notice that average of  $\nabla u$  is nul) yields

$$\frac{b}{2} \|\nabla\Pi\|_{L^2} \lesssim \|u\|_{L^3} \|\nabla^2 u\|_{L^2} + \|\Delta u \cdot \nabla S_m a\|_{\dot{H}^{-1}} + \|a\|_{L^\infty} \|\Delta u\|_{L^2}.$$

On the other hand, we prove easily thanks to the divergence free condition that

$$\|\Delta u \cdot \nabla S_m a\|_{\dot{H}^{-1}} \leq C \|a\|_{L^\infty} \|\Delta u\|_{L^2}.$$

Despite the fact that average of  $u$  is not nul, we have  $\|u\|_{L^3} \leq C(\rho_0) \|u\|_{B_{2,1}^{\frac{1}{2}}}$ . Hence, one has

$$(57) \quad \frac{b}{2} \|\nabla\Pi\|_{L_t^1(L^2)} \lesssim (\|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + 2\|a\|_{L_t^\infty(L^\infty)}) \|u\|_{L_t^1(\dot{H}^2)}.$$

Plugging (57) in the estimate (55), we finally get an estimate of the pressure, in which the right-hand side is independent of the pressure: we got rid of the term  $\|\nabla\Pi\|_{L^2}$ . Indeed, (55) becomes

$$(58) \quad \begin{aligned} \frac{b}{2} \|\nabla\Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} &\lesssim \int_0^t \|u\|_{B_{2,1}^{\frac{1}{2}}} \|\nabla u\|_{L^\infty} dt' + \|(S_m a - a)\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})} \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} \\ &\quad + 2^{2m} \|a\|_{L_t^\infty(L^2)} \|u\|_{L_t^1(\dot{H}^2)} \left(1 + \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + \|a\|_{L_t^\infty(L^\infty)}\right). \end{aligned}$$

Plugging (57) in the estimate (51), we also get

$$(59) \quad \begin{aligned} \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + C b \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} &\leq \|u_0\|_{B_{2,1}^{\frac{1}{2}}} + \|a - S_m a\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})} \left( \|\nabla\Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} + \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} \right) \\ &\quad + \int_0^t \|\nabla u(t')\|_{L^\infty} \|u(t')\|_{B_{2,1}^{\frac{1}{2}}} dt' + 2^m \|a\|_{L_t^\infty(L^\infty)} \|u\|_{L_t^1(B_{2,1}^{\frac{3}{2}})} \\ &\quad + 2^{2m+1} \|a\|_{L_t^\infty(L^2)} \|u\|_{L_t^1(\dot{H}^2)} \left(1 + \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + 2\|a\|_{L_t^\infty(L^\infty)}\right), \end{aligned}$$

Summing (59) with (58) and using obvious estimates on the transport equation below

$$\|a\|_{L_t^\infty(L^\infty)} \leq \|a_0\|_{L^\infty} \quad \text{and} \quad \|a\|_{L_t^\infty(L^2)} \leq \|a_0\|_{L^2},$$

leads to

$$\begin{aligned}
& \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + C b \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} + \frac{b}{2} \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} \lesssim \|u_0\|_{B_{2,1}^{\frac{1}{2}}} \\
& + \|a - S_m a\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})} (\|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} + \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})}) \\
& + \int_0^t \|\nabla u(t')\|_{L^\infty} \|u(t')\|_{B_{2,1}^{\frac{1}{2}}} dt' + 2^m \|a_0\|_{L^\infty} \|u\|_{L_t^1(B_{2,1}^{\frac{3}{2}})} \\
& + 2^{2m+1} \|a_0\|_{L^2} \|u\|_{L_t^1(\dot{H}^2)} (1 + \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + \|a_0\|_{L^\infty}) \\
& + \|u\|_{L_t^1(\dot{H}^2)} (\|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + 2 \|a_0\|_{L^\infty}).
\end{aligned}$$

Once again, the smallness condition simplifies the above estimate

$$\begin{aligned}
(60) \quad & \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + C b \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} + \frac{b}{2} \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} \lesssim \|u_0\|_{B_{2,1}^{\frac{1}{2}}} + 2 \int_0^t \|u(t')\|_{B_{2,1}^{\frac{5}{2}}} \|u(t')\|_{B_{2,1}^{\frac{1}{2}}} dt' \\
& + (1 + 2^{2m+1} \|a_0\|_{L^2}) \|u\|_{L_t^1(\dot{H}^2)} (1 + \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + \|a_0\|_{L^\infty}) \\
& + 2^m \|a_0\|_{L^\infty} \|u\|_{L_t^1(B_{2,1}^{\frac{3}{2}})}.
\end{aligned}$$

Let us recall some interpolation properties. The following inequalities hold on the torus:

$$\|u\|_{\dot{H}^2} \leq C \|u\|_{B_{2,1}^{\frac{1}{2}}}^{\frac{1}{4}} \|u\|_{B_{2,1}^{\frac{5}{2}}}^{\frac{3}{4}} \quad \text{and} \quad \|u\|_{B_{2,1}^{\frac{3}{2}}} \leq C \|u\|_{B_{2,1}^{\frac{1}{2}}}^{\frac{1}{2}} \|u\|_{B_{2,1}^{\frac{5}{2}}}^{\frac{1}{2}}.$$

They are due the product laws in Besov spaces (cf Lemma 6.2). For instance, the first one stems from

$$\|u\|_{\dot{H}^2} = \|\nabla u\|_{\dot{H}^1} \leq \|\nabla u\|_{H^1} \leq \|\nabla u\|_{B_{2,1}^1} \leq C \|\nabla u\|_{B_{2,1}^{-\frac{1}{2}}}^{\frac{1}{4}} \|\nabla u\|_{B_{2,1}^{\frac{3}{2}}}^{\frac{3}{4}}.$$

Obviously, by integration in time and thanks to Hölder's inequality, we have

$$\|u\|_{L_t^1(\dot{H}^2)} \leq C \|u\|_{L_t^1(B_{2,1}^{\frac{1}{2}})}^{\frac{1}{4}} \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})}^{\frac{3}{4}} \quad \text{and} \quad \|u\|_{L_t^1(B_{2,1}^{\frac{3}{2}})} \leq C \|u\|_{L_t^1(B_{2,1}^{\frac{1}{2}})}^{\frac{1}{2}} \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})}^{\frac{1}{2}}.$$

By virtue of Young's inequalities

$$xy \leq \frac{x^4}{4} + \frac{3y^{\frac{4}{3}}}{4} \quad \text{and} \quad xy \leq \frac{x^2}{2} + \frac{y^2}{2},$$

Estimate (60) becomes

$$\begin{aligned}
& \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + C b \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} + \frac{b}{2} \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} \lesssim \|u_0\|_{B_{2,1}^{\frac{1}{2}}} + 2 \int_0^t \|\nabla u(t')\|_{L^\infty} \|u(t')\|_{B_{2,1}^{\frac{1}{2}}} dt' \\
& + (1 + 2^{8m} \|a_0\|_{L^2}^4) \|u\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} (1 + \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})}^4 + \|a_0\|_{L^\infty}^4) + \frac{b}{4} \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} \\
& + 2^{2m} \|a_0\|_{L^\infty}^2 \|u\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} + \frac{b}{4} \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})}.
\end{aligned}$$

which can be simplified by

$$\begin{aligned}
& \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + C \frac{b}{2} \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} + \frac{b}{2} \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} \lesssim \|u_0\|_{B_{2,1}^{\frac{1}{2}}} + 2 \int_0^t \|\nabla u(t')\|_{L^\infty} \|u(t')\|_{B_{2,1}^{\frac{1}{2}}} dt' \\
& + (1 + 2^{8m} \|a_0\|_{L^2}^4) \|u\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} (1 + \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})}^4 + \|a_0\|_{L^\infty}^4) \\
& + 2^{2m} \|a_0\|_{L^\infty}^2 \|u\|_{L_t^1(B_{2,1}^{\frac{1}{2}})}.
\end{aligned}$$



This concludes the proof of Lemma 1.3.

*Continuation of the proof of existence part of Theorem 1.2.* This stems from the obvious fact :  $B_{2,1}^{\frac{3}{2}} \hookrightarrow L^\infty$  and thus

$$\|\nabla u\|_{L^\infty} \leq \|\nabla u\|_{B_{2,1}^{\frac{3}{2}}}.$$

Therefore, we get

$$\begin{aligned} \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + C b \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} + \frac{b}{2} \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} &\lesssim \|u_0\|_{B_{2,1}^{\frac{1}{2}}} + 2 \int_0^t \|u(t')\|_{B_{2,1}^{\frac{5}{2}}} \|ut(t')\|_{B_{2,1}^{\frac{1}{2}}} dt' \\ &+ (1 + 2^{8m} \|a_0\|_{L^2}^4) \|u\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} (1 + \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})}^4 + \|a_0\|_{L^\infty}^4) \\ &+ 2^{2m} \|a_0\|_{L^\infty}^2 \|u\|_{L_t^1(B_{2,1}^{\frac{1}{2}})}. \end{aligned}$$

As a result, we get

(61)

$$\begin{aligned} \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + C \frac{b}{2} \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} + \frac{b}{2} \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} &\lesssim \|u_0\|_{B_{2,1}^{\frac{1}{2}}} + 2 \int_0^t \|u\|_{B_{2,1}^{\frac{5}{2}}} \|u\|_{B_{2,1}^{\frac{1}{2}}} dt' \\ &+ \int_0^t \|u(t')\|_{B_{2,1}^{\frac{1}{2}}} \left( 2^{2m} \|a_0\|_{L^\infty}^2 + (1 + 2^{8m} \|a_0\|_{L^2}^4) (1 + \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})}^4 + \|a_0\|_{L^\infty}^4) \right) dt'. \end{aligned}$$

Let  $\varepsilon_0 > 0$ . Let us introduce the time  $T_0$  such that

$$T_0 \stackrel{\text{def}}{=} \sup\{0 \leq t \leq T^* \mid \|u(t)\|_{B_{2,1}^{\frac{1}{2}}} \leq \varepsilon_0\}.$$

Hence, for any  $t \leq T_0$ , we have

$$\begin{aligned} \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + C \frac{b}{2} \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} + \frac{b}{2} \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} &\lesssim \|u_0\|_{B_{2,1}^{\frac{1}{2}}} + 2\varepsilon_0 \int_0^t \|u(t')\|_{B_{2,1}^{\frac{5}{2}}} dt' \\ &+ \int_0^t \|u(t')\|_{B_{2,1}^{\frac{1}{2}}} \left( 2^{2m} \|a_0\|_{L^\infty}^2 + (1 + 2^{8m} \|a_0\|_{L^2}^4) (1 + \varepsilon_0^4 + \|a_0\|_{L^\infty}^4) \right) dt'. \end{aligned}$$

Choosing  $\varepsilon_0$  small enough, namely  $\varepsilon_0 \leq \frac{Cb}{4}$ , Gronwall lemma implies that for any  $t \leq T_0$ ,

(62)

$$\begin{aligned} \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + \frac{Cb}{4} \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} + \frac{b}{2} \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} &\lesssim \|u_0\|_{B_{2,1}^{\frac{1}{2}}} \\ &\times \exp \left( (T_0 (2^{2m} \|a_0\|_{L^\infty}^2 + (1 + 2^{8m} \|a_0\|_{L^2}^4) (1 + \left(\frac{b}{4}\right)^4 + \|a_0\|_{L^\infty}^4)) \right). \end{aligned}$$

As a result, we get the a priori estima on the velocity

$$(63) \quad \text{For any } t \leq T_0, \quad \|u\|_{L_t^\infty(B_{2,1}^{\frac{1}{2}})} + \|u\|_{L_t^1(B_{2,1}^{\frac{5}{2}})} + \frac{b}{2} \|\nabla \Pi\|_{L_t^1(B_{2,1}^{\frac{1}{2}})} \leq C \|u_0\|_{B_{2,1}^{\frac{1}{2}}}.$$

This concludes the proof of (8) : until the (small) time  $T_0$ , the solution is controlled by initial data, up to a multiplicative constant. This ends up the proof of the local-existence part of Theorem 1.2.

**4.2. Uniqueness part.** The uniqueness part has been already done in [3]. We refer the reader to it for more details. Let us recall some details. Let  $(a_1, u_1, \nabla \Pi_1)$  and  $(a_2, u_2, \nabla \Pi_2)$  be two solutions of the system (4), satisfying the smallness hypothesis  $\|a - S_m a\|_{B_{2,1}^{\frac{3}{2}}} \leq c$  and such that

$$(64) \quad (a_i, u_i, \nabla \Pi_i) \in \mathcal{C}([0, T], B_{2,1}^{\frac{3}{2}}) \times \mathcal{C}([0, T], B_{2,1}^{\frac{1}{2}}) \cap L^1([0, T], B_{2,1}^{\frac{5}{2}}) \times L^1([0, T], B_{2,1}^{\frac{1}{2}}).$$

We define as one expects

$$(\delta a, \delta u, \nabla \delta \Pi) \stackrel{\text{def}}{=} (a_2 - a_1, u_2 - u_1, \nabla \Pi_2 - \nabla \Pi_1),$$

so that  $(\delta a, \delta u, \nabla \delta \Pi)$  solves the following system

$$(65) \quad \begin{cases} \partial_t \delta a + u_2 \cdot \nabla \delta a & = & -\delta u \cdot \nabla a_1 \\ \partial_t \delta u + u_2 \cdot \nabla \delta u - (1 + a_2)(-\nabla \delta \Pi - \Delta \delta u) & = & -\delta u \cdot \nabla u_1 + \delta a(\Delta u_1 - \nabla \Pi_1) \\ \operatorname{div} \delta u & = & 0 \\ (\delta a, \delta u)|_{t=0} & = & (0, 0). \end{cases}$$

We prove that such solution of this system satisfies

$$(66) \quad (\delta a, \delta u, \nabla \delta \Pi) \in \mathcal{C}([0, T], B_{2,1}^{\frac{3}{2}}) \times \mathcal{C}([0, T], B_{2,1}^{-\frac{1}{2}}) \cap L^1([0, T], B_{2,1}^{\frac{3}{2}}) \times L^1([0, T], B_{2,1}^{-\frac{1}{2}}).$$

**Remark 4.1.** Notice that, owing to the presence of a transport equation, we loose one derivative in the estimate involving  $\delta a$ .

## 5. PROOF OF THE GLOBAL WELLPOSEDNESS PART OF THE MAIN THEOREM

This section is devoted to the proof of Theorem 1.4, which provides the global property of the main Theorem 1.1.

$$(67) \quad \begin{cases} \partial_t \rho + u \cdot \nabla \rho & = & 0 \\ \rho(\partial_t u + u \cdot \nabla u) - \Delta u + \nabla \Pi & = & 0 \\ \operatorname{div} u & = & 0 \\ (\rho, u)|_{t=0} & = & (\rho_0, u_0). \end{cases}$$

In a sake of simplicity, we skip the regularisation process (Friedrich methods) and we only present the a priori estimates for smooth enough solution  $(\rho, u)$ , which provide the existence part of Theorem 1.4. Concerning the uniqueness part, we refer the reader to the paper of M. Paicu, P. Zhang and Z. Zhang (see [17]). We underline that Lagrangian coordinates are necessary to prove the uniqueness, owing to the very low regularity hypothesis on the density (which is only supposed to be bounded from above and from below). Let us proceed firstly to an  $L^2$ -energy estimate, which leads to the result on  $B_0$ . Then we will get estimate on  $B_1$ , thanks to an  $H^1$ -energy estimate.

• Proof of (12). Taking the  $L^2$  inner product of momentum equation with  $u$  in the system (67), we get :

$$(\rho(\partial_t u + u \cdot \nabla u) | u)_{L^2} - (\Delta u | u)_{L^2} + 0 = 0.$$

We check that  $(\rho(\partial_t u + u \cdot \nabla u) | u)_{L^2} = \frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} u\|_{L^2}^2$ .

This stems from the computations below

$$\begin{aligned} (\rho(\partial_t u + u \cdot \nabla u) | u)_{L^2} &= \frac{1}{2} \int_{\mathbb{T}^3} \rho \partial_t |u|^2 dx + \frac{1}{2} \int_{\mathbb{T}^3} \rho u \cdot \nabla |u|^2 dx \\ &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} \rho |u|^2 - \frac{1}{2} \int_{\mathbb{T}^3} \partial_t \rho |u|^2 dx + \frac{1}{2} \int_{\mathbb{T}^3} \rho u \cdot \nabla |u|^2 dx. \end{aligned}$$

However,  $\int_{\mathbb{T}^3} \rho u \cdot \nabla |u|^2 = - \int_{\mathbb{T}^3} (u \cdot \nabla \rho) |u|^2$ . Therefore, the transport equation yields

$$\begin{aligned} (\rho(\partial_t u + u \cdot \nabla u), u)_{L^2} &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} \rho |u|^2 - \frac{1}{2} \int_{\mathbb{T}^3} (\partial_t \rho + u \cdot \nabla \rho) |u|^2 \\ &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} \rho |u|^2. \end{aligned}$$

Finally, an integration in time provides the desired estimate

$$(68) \quad \frac{1}{2} \|\sqrt{\rho}u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(t')\|_{L^2}^2 dt' = \frac{1}{2} \|\sqrt{\rho_0}u_0\|_{L^2}^2.$$

This concludes the proof of (12). Now let us proceed to the proof of (13).

• Proof of (13). The idea is the same as the previous one : we take the  $L^2$  inner product of momentum equation with  $\partial_t u$  in the system (67), we get :

$$(\sqrt{\rho} \partial_t u | \sqrt{\rho} \partial_t u)_{L^2} + (\sqrt{\rho} u \cdot \nabla u | \sqrt{\rho} \partial_t u)_{L^2} + \frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 = 0,$$

which leads to

$$(69) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \|\sqrt{\rho} \partial_t u(t)\|_{L^2}^2 &\leq \|\sqrt{\rho} u \cdot \nabla u(t)\|_{L^2} \|\sqrt{\rho} \partial_t u(t)\|_{L^2} \\ &\leq \|\sqrt{\rho} u(t)\|_{L^6} \|\nabla u(t)\|_{L^3} \|\sqrt{\rho} \partial_t u(t)\|_{L^2} \end{aligned}$$

Applying Proposition 2.5 on the term  $\|u(t)\|_{L^6}$  and Proposition 2.3 on the term  $\|\nabla u(t)\|_{L^3}$  gives rise to

$$(70) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \|\sqrt{\rho} \partial_t u(t)\|_{L^2}^2 &\leq C(\rho_0) \|\nabla u(t)\|_{L^2} \|\nabla u(t)\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u(t)\|_{L^2}^{\frac{1}{2}} \|\sqrt{\rho} \partial_t u(t)\|_{L^2} \\ &\leq C(\rho_0) \|\nabla u(t)\|_{L^2}^{\frac{3}{2}} \|\nabla^2 u(t)\|_{L^2}^{\frac{1}{2}} \|\sqrt{\rho} \partial_t u(t)\|_{L^2}. \end{aligned}$$

Then, Young inequality yields

$$(71) \quad \frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \frac{1}{2} \|\sqrt{\rho} \partial_t u(t)\|_{L^2}^2 \leq \frac{1}{2} C(\rho_0) \|\nabla u(t)\|_{L^2}^3 \|\nabla^2 u(t)\|_{L^2}.$$

We have to estimate the term  $\|\nabla^2 u\|_{L^2}$ . Applying the  $L^2$ -norm in the momentum equation, we get

$$\|\nabla^2 u(t)\|_{L^2} + \|\nabla \Pi(t)\|_{L^2} \leq \|\rho(t)\|_{L^\infty}^{\frac{1}{2}} \left( \|\sqrt{\rho} \partial_t u(t)\|_{L^2} + \|\sqrt{\rho} u(t)\|_{L^6} \|\nabla u\|_{L^3} \right).$$

Once again, by virtue of Proposition 2.5 and Gagliardo-Nirenberg inequality, one has

$$\|\nabla^2 u\|_{L^2} + \|\nabla \Pi\|_{L^2} \leq C(\rho_0) \left( \|\sqrt{\rho} \partial_t u(t)\|_{L^2} + \|\nabla u(t)\|_{L^2} \|\nabla u(t)\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u(t)\|_{L^2}^{\frac{1}{2}} \right).$$

Young inequality implies

$$(72) \quad \frac{1}{2} \|\nabla^2 u(t)\|_{L^2} + \|\nabla \Pi\|_{L^2} \leq C(\rho_0) \|\sqrt{\rho} \partial_t u(t)\|_{L^2} + \frac{1}{2} \|\nabla u(t)\|_{L^2}^3.$$

Plugging Inequality (72) in (71) and applying Young inequality gives

$$(73) \quad \frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \frac{1}{2} \|\sqrt{\rho} \partial_t u(t)\|_{L^2}^2 \leq C(\rho_0) \|\nabla u(t)\|_{L^2}^6 + \frac{1}{4} \|\sqrt{\rho} \partial_t u(t)\|_{L^2}^2.$$

As a result, we have :

$$(74) \quad \frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \frac{1}{4} \|\sqrt{\rho} \partial_t u(t)\|_{L^2}^2 \leq C(\rho_0) \|\nabla u(t)\|_{L^2}^6.$$

We sum (74) and (72) and we get :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \frac{1}{4} \|\sqrt{\rho} \partial_t u(t)\|_{L^2}^2 + \|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla \Pi(t)\|_{L^2}^2 &\leq C(\rho_0) \left( \|\nabla u(t)\|_{L^2}^6 \right. \\ &\quad \left. + \frac{1}{8} \|\sqrt{\rho} \partial_t u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^6 \right). \end{aligned}$$

Finally, we have by integration in time

$$(75) \quad \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \int_0^t \left( \frac{1}{8} \|\sqrt{\rho} \partial_t u(t')\|_{L^2}^2 + \|\nabla^2 u(t')\|_{L^2}^2 + \|\nabla \Pi(t')\|_{L^2}^2 \right) dt' \leq \frac{1}{2} \|\nabla u_0\|_{L^2}^2 + C(\rho_0) \int_0^t \|\nabla u(t')\|_{L^2}^6 dt'.$$

Let us focus for a while on the term  $\int_0^t \|\nabla u(t')\|_{L^2}^6 dt'$ . It seems clear that

$$\int_0^t \|\nabla u(t')\|_{L^2}^6 dt' \leq \|\nabla u\|_{L_t^\infty(L^2)}^4 \int_0^t \|\nabla u(t')\|_{L^2}^2 dt',$$

which leads to, by virtue of (13) and definition of  $B_1$

$$\int_0^t \|\nabla u(t')\|_{L^2}^6 dt' \leq \|u_0\|_{L^2}^2 B_1^2(t).$$

Finally, we get

$$B_1(t) \leq \frac{1}{2} \|\nabla u_0\|_{L^2}^2 + C(\rho_0) \|u_0\|_{L^2}^2 B_1^2(t).$$

As long as the smallness condition on  $u_0$  is satisfied, we obtain Estimate (13), which conclude the proof of this estimate.

• Proof of (14). Firstly, we derive the momentum equations, with respect to the time  $t$ . Then, we take the  $L^2$  inner product with  $\partial_t u$ .

The derivated momentum equation is given by the following formula :

$$\begin{aligned} (\rho \partial_{tt} u | \partial_t u)_{L^2} - (\Delta \partial_t u | \partial_t u)_{L^2} &= -(\partial_t \rho (\partial_t u + u \cdot \nabla u) | \partial_t u)_{L^2} - (\rho \partial_t u \cdot \nabla u | \partial_t u)_{L^2} \\ &\quad - (\rho u \cdot \nabla \partial_t u | \partial_t u)_{L^2}. \end{aligned}$$

By hypothesis on the density, the left-hand side can be bounded from below by :

$$\begin{aligned} \frac{m}{2} \frac{d}{dt} (\|\partial_t u\|_{L^2}^2) + \|\nabla \partial_t u\|_{L^2}^2 &\leq \frac{m}{2} \|\partial_t u\|_{L^2}^2 - (\rho \partial_t u \cdot \nabla u | \partial_t u)_{L^2} - (\rho u \cdot \nabla \partial_t u | \partial_t u)_{L^2} \\ &\quad - (\partial_t \rho \partial_t u | \partial_t u)_{L^2} - (\partial_t \rho u \cdot \nabla u | \partial_t u)_{L^2}. \end{aligned}$$

Let us point out that  $(\rho u \cdot \nabla \partial_t u | \partial_t u)_{L^2}$  is in fact nul, by virtue of the divergence free condition. Taking the modulus, applying triangular inequality and finally, using the mass equation on the density:

$$(76) \quad \begin{aligned} \frac{m}{2} \frac{d}{dt} (\|\partial_t u\|_{L^2}^2) + \|\nabla \partial_t u\|_{L^2}^2 &\leq \frac{m}{2} \|\partial_t u\|_{L^2}^2 + \int_{\mathbb{T}^3} \left| \rho (\partial_t u \cdot \nabla u) \partial_t u \right| dx \\ &\quad + \left| (\operatorname{div}(\rho u) | (\partial_t u)^2)_{L^2} \right| + \left| (\operatorname{div}(\rho u) u \cdot \nabla u) | \partial_t u \right|_{L^2} \\ &\leq \sum_{k=1}^6 I_k(t), \end{aligned}$$

with

$$\begin{aligned}
I_1(t) &= \frac{m}{2} \|\partial_t u\|_{L^2}^2 dx, \\
I_2(t) &= \int_{\mathbb{T}^3} \left| \rho (\partial_t u \cdot \nabla u) \partial_t u \right| dx, \\
I_3(t) &= 2 \int_{\mathbb{T}^3} \left| \rho u \nabla(\partial_t u) \partial_t u \right| dx, \\
I_4(t) &= \int_{\mathbb{T}^3} \left| \rho ((u \cdot \nabla u) \cdot \nabla u) \cdot \partial_t u \right| dx, \\
I_5(t) &= \int_{\mathbb{T}^3} \left| \rho ((u \otimes u) : \nabla^2 u) \cdot \partial_t u \right| dx, \\
I_6(t) &= \int_{\mathbb{T}^3} \left| \rho (u \cdot \nabla u) \cdot (u \cdot \nabla(\partial_t u)) \right| dx.
\end{aligned} \tag{77}$$

As far as  $I_2(t)$  is concerned, firstly we apply Hölder's inequality and we get

$$\begin{aligned}
I_2(t) &= \int_{\mathbb{T}^3} \left| \rho (\partial_t u \cdot \nabla u) \partial_t u \right| dx \\
&\leq M \|\partial_t u(t)\|_{L^2} \|\partial_t u(t)\|_{L^6} \|\nabla u(t)\|_{L^3}.
\end{aligned} \tag{78}$$

Once again, classical Sobolev embedding can not be applied directly to the term  $\|\partial_t u(t)\|_{L^6}$ . We shall consider the term  $\overline{\partial_t u(t)}$  and adapt Lemma 2.4. Firstly, notice that  $\int_{\mathbb{T}^3} \rho(t, x) \partial_t u(t, x) dx = 0$ , due to an integration of the momentum equation in (67)). Hence, *the average method* gives rise to the following computation

$$\int_{\mathbb{T}^3} (\rho(t, x) - \bar{\rho}(t)) (\partial_t u(t, x) - \overline{\partial_t u(t)}) dx = \int_{\mathbb{T}^3} \rho(t, x) \partial_t u(t, x) dx - \bar{\rho}(t) \overline{\partial_t u(t)}.$$

By virtue of remarks 1.1 and 1.3, one has

$$|\overline{\partial_t u(t)}| \leq \frac{1}{\bar{\rho}_0} \|\rho_0 - \bar{\rho}_0\|_{L^2} \|\partial_t u(t) - \overline{\partial_t u(t)}\|_{L^2},$$

which gives, thanks to Poincaré-Wirtinger

$$|\overline{\partial_t u(t)}| \leq \frac{1}{\bar{\rho}_0} \|\rho_0 - \bar{\rho}_0\|_{L^2} \|\nabla \partial_t u(t)\|_{L^2}.$$

Therefore, we deduce from the above computation that

$$\|\partial_t u(t)\|_{L^6} \leq \|\partial_t u(t) - \overline{\partial_t u(t)}\|_{L^6} + |\overline{\partial_t u(t)}| \leq C(\rho_0) \|\nabla \partial_t u(t)\|_{L^2}.$$

Thanks to Gagliardo-Nirenberg and Young inequalities, we infer that

$$\begin{aligned}
I_2(t) &\leq C(\rho_0) \|\partial_t u(t)\|_{L^2} \|\nabla \partial_t u(t)\|_{L^2} \|\nabla u(t)\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u(t)\|_{L^2}^{\frac{1}{2}} \\
&\leq C(\rho_0) \|\partial_t u(t)\|_{L^2}^2 \|\nabla u(t)\|_{L^2} \|\nabla^2 u(t)\|_{L^2} + \frac{1}{4} \|\nabla \partial_t u(t)\|_{L^2}^2 \\
&\leq C(\rho_0) \|\partial_t u(t)\|_{L^2}^2 \left( \|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 u(t)\|_{L^2}^2 \right) + \frac{1}{4} \|\nabla \partial_t u(t)\|_{L^2}^2.
\end{aligned} \tag{79}$$

Concerning estimate of  $I_3(t)$ , we get

$$\begin{aligned}
I_3(t) &= \int_{\mathbb{T}^3} \left| \rho u \nabla(\partial_t u(t)) \partial_t u(t) \right| dx \\
&\leq M \|u \partial_t u\|_{L^2} \|\nabla \partial_t u\|_{L^2} \\
&\leq M \|u\|_{L^3} \|\partial_t u\|_{L^6} \|\nabla \partial_t u\|_{L^2}.
\end{aligned} \tag{80}$$

Applying *the average method* for  $\|\partial_t u(t)\|_{L^6}$  and  $\|u(t)\|_{L^3}$ , we infer that

$$(81) \quad \begin{aligned} I_3(t) &\leq C(\rho_0) \|u(t)\|_{L^3} \|\nabla \partial_t u(t)\|_{L^2}^2 \\ &\leq C(\rho_0) \|\nabla u(t)\|_{L^2} \|\nabla \partial_t u(t)\|_{L^2}^2. \end{aligned}$$

Concerning  $I_4(t)$ ,  $I_5(t)$ , and  $I_6(t)$ , previous computations hold (applying Proposition 2.5 and Young inequality) :

$$(82) \quad \begin{aligned} I_4(t) &= \int_{\mathbb{T}^3} \left| \rho((u \cdot \nabla u) \cdot \nabla u) \cdot \partial_t u \right| dx \\ &\leq M \|u(t)\|_{L^6} \|\nabla u(t)\|_{L^6}^2 \|\partial_t u(t)\|_{L^2} \\ &\leq C(\rho_0) \|\nabla u(t)\|_{L^2} \|\nabla^2 u(t)\|_{L^2}^2 \|\partial_t u(t)\|_{L^2} \\ &\leq \frac{1}{4} C(\rho_0) \left( \|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 u(t)\|_{L^2}^2 \right) \left( \|\nabla^2 u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2 \right). \end{aligned}$$

$$(83) \quad \begin{aligned} I_5(t) &= \int_{\mathbb{T}^3} \left| \rho((u \otimes u) : \nabla^2 u) \cdot \partial_t u \right| dx \\ &\leq M \|u^2 \partial_t u(t)\|_{L^2} \|\nabla^2 u(t)\|_{L^2} \\ &\leq C(\rho_0) \|u(t)\|_{L^6}^2 \|\nabla \partial_t u(t)\|_{L^2} \|\nabla^2 u(t)\|_{L^2} \\ &\leq C(\rho_0) \|\nabla u(t)\|_{L^2}^2 \|\nabla \partial_t u(t)\|_{L^2} \|\nabla^2 u(t)\|_{L^2} \\ &\leq C(\rho_0) \|\nabla u(t)\|_{L^2}^4 \|\nabla^2 u(t)\|_{L^2}^2 + \frac{1}{4} \|\nabla \partial_t u(t)\|_{L^2}^2. \end{aligned}$$

Similar computation holds for the last term  $I_6(t)$ .

$$(84) \quad \begin{aligned} I_6(t) &= \int_{\mathbb{T}^3} \left| \rho(u \cdot \nabla u) \cdot (u \cdot \nabla(\partial_t u)) \right| dx \\ &\leq M \|u^2 \nabla u(t)\|_{L^2} \|\nabla \partial_t u(t)\|_{L^2} \\ &\leq C(\rho_0) \|\nabla u(t)\|_{L^2}^4 \|\nabla^2 u(t)\|_{L^2}^2 + \frac{1}{4} \|\nabla \partial_t u(t)\|_{L^2}^2. \end{aligned}$$

Let us keep on the proof. Plugging these above estimates into the (76) gives rise to

$$(85) \quad \begin{aligned} \frac{m}{2} \frac{d}{dt} (\|\partial_t u\|_{L^2}^2) + \|\nabla \partial_t u(t)\|_{L^2}^2 &\leq \frac{m}{2} \|\partial_t u(t)\|_{L^2}^2 + C(\rho_0) \|\partial_t u(t)\|_{L^2}^2 \left( \|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 u(t)\|_{L^2}^2 \right) \\ &\quad + \frac{1}{4} \|\nabla \partial_t u(t)\|_{L^2}^2 + C(\rho_0) \|\nabla u(t)\|_{L^2} \|\nabla \partial_t u(t)\|_{L^2}^2 \\ &\quad + C(\rho_0) \left( \|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 u(t)\|_{L^2}^2 \right) \left( \|\nabla^2 u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2 \right) \\ &\quad + 2 C(\rho_0) \|\nabla u(t)\|_{L^2}^4 \|\nabla^2 u(t)\|_{L^2}^2 + \frac{1}{2} \|\nabla \partial_t u(t)\|_{L^2}^2, \end{aligned}$$

so that

$$\begin{aligned} \frac{m}{2} \frac{d}{dt} (\|\partial_t u(t)\|_{L^2}^2) + \frac{1}{4} \|\nabla \partial_t u(t)\|_{L^2}^2 &\leq \frac{m}{2} \|\partial_t u(t)\|_{L^2}^2 + C(\rho_0) \|\nabla u(t)\|_{L^2} \|\nabla \partial_t u(t)\|_{L^2}^2 \\ &\quad + 2 C(\rho_0) \|\nabla u(t)\|_{L^2}^4 \|\nabla^2 u(t)\|_{L^2}^2 \\ &\quad + C(\rho_0) \left( \|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 u(t)\|_{L^2}^2 \right) \left( \|\nabla^2 u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2 \right). \end{aligned}$$

By integration in time, we have :

$$\begin{aligned}
(86) \quad & \frac{m}{2} \|\partial_t u(t)\|_{L^2}^2 + \frac{1}{2} \int_0^t \|\nabla \partial_t u(t')\|_{L^2}^2 dt' \leq \|u_0\|_{H^2}^2 + \frac{m}{2} \int_0^t \|\partial_t u(t')\|_{L^2}^2 dt' \\
& + C(\rho_0) \int_0^t \|\nabla u(t')\|_{L^2} \|\nabla \partial_t u(t')\|_{L^2}^2 dt' \\
& + C(\rho_0) \int_0^t \|\nabla u(t')\|_{L^2}^4 \|\nabla^2 u(t')\|_{L^2}^2 dt' \\
& + C(\rho_0) \int_0^t \left( \|\nabla u(t')\|_{L^2}^2 + \|\nabla^2 u(t')\|_{L^2}^2 \right) \left( \|\nabla^2 u(t')\|_{L^2}^2 + \|\partial_t u(t')\|_{L^2}^2 \right) dt'.
\end{aligned}$$

Concerning the term  $\int_0^t \|\nabla u(t')\|_{L^2} \|\nabla \partial_t u(t')\|_{L^2}^2 dt'$

$$\int_0^t \|\nabla u(t')\|_{L^2} \|\nabla \partial_t u(t')\|_{L^2}^2 dt' \leq \|\nabla u\|_{L_T^\infty(L^2)} \int_0^t \|\nabla \partial_t u(t')\|_{L^2}^2 dt',$$

which becomes, by virtue of Theorem 1.4,

$$\int_0^t \|\nabla u(t')\|_{L^2} \|\nabla \partial_t u(t')\|_{L^2}^2 dt' \leq C \|\nabla u_0\|_{L^2} \int_0^t \|\nabla \partial_t u(t')\|_{L^2}^2 dt'.$$

Same argument combining with Theorem 1.4 gives rise to

$$\int_0^t \|\nabla u(t')\|_{L^2}^4 \|\nabla^2 u(t')\|_{L^2}^2 dt' \leq C \|\nabla u_0\|_{L^2}^6.$$

As a result, Inequation (86) can be rewritten as follows ( providing we choose  $\|\nabla u_0\|_{L^2}$  small enough)

$$\begin{aligned}
(87) \quad & \frac{m}{2} \|\partial_t u(t)\|_{L^2}^2 + \frac{1}{3} \int_0^t \|\nabla \partial_t u(t')\|_{L^2}^2 dt' \lesssim \|u_0\|_{H^2}^2 + \frac{m}{2} \|\nabla u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^6 \\
& + \int_0^t \left( \|\nabla u(t')\|_{L^2}^2 + \|\nabla^2 u(t')\|_{L^2}^2 \right) \left( \|\nabla^2 u(t')\|_{L^2}^2 + \|\partial_t u(t')\|_{L^2}^2 \right) dt'.
\end{aligned}$$

Moreover, the momentum equation given by

$$-\Delta u + \nabla \Pi = -\rho (\partial_t u + u \cdot \nabla u),$$

which along with the classical estimates on the Stokes system, ensures that

$$\begin{aligned}
\|\nabla^2 u(t)\|_{L^2} + \|\nabla \Pi(t)\|_{L^2} & \leq C \left( \|\partial_t u(t)\|_{L^2} + \|\nabla u(t)\|_{L^2}^{\frac{3}{2}} \|\nabla^2 u(t)\|_{L^2}^{\frac{1}{2}} \right) \\
& \lesssim \|\partial_t u(t)\|_{L^2} + \|\nabla u(t)\|_{L^2}^3 + \frac{1}{2} \|\nabla^2 u(t)\|_{L^2}.
\end{aligned}$$

So that, we get

$$(88) \quad \frac{1}{2} \|\nabla^2 u(t)\|_{L^2} + \|\nabla \Pi(t)\|_{L^2} \lesssim \|\partial_t u(t)\|_{L^2} + \|\nabla u(t)\|_{L^2}^3.$$

By virtue of Theorem 1.4, we obtain

$$(89) \quad \sup_{t \in [0, T]} \left( \frac{1}{2} \|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla \Pi(t)\|_{L^2}^2 \right) \lesssim \sup_{t \in [0, T]} \left( \|\partial_t u(t)\|_{L^2}^2 \right) + \|\nabla u_0\|_{L^2}^6.$$

**Remark 5.1.** Let us point out that searching an estimate of  $\|u\|_{L_T^2(H^3)}$  is a natural idea here since the initial velocity  $u_0$  belongs to the space  $H^2$ . But actually, it is not relevant. Indeed, to perform it, we shall use the theory of Stokes problems. We shall begin derivating the momentum equation with respect to the space, and then, we shall take the  $L^2$  norm. But, such an approach is doomed to fail, because requires an estimate on  $\sup_{t \in [0, T]} \|\nabla \rho\|_{L^\infty}$ , which is not our case here, since the density function only belongs to  $L^\infty([0, T] \times \mathbb{T}^3)$ .

Once again, the momentum equation gives

$$-\Delta u + \nabla \Pi = -\rho(\partial_t u + u \cdot \nabla u).$$

We take the  $L^6$ -norm and use the fact that  $\|u \cdot \nabla u(t)\|_{L^6} \leq C \|\nabla(u \cdot \nabla u(t))\|_{L^2}$  since  $\overline{u \cdot \nabla u} = 0$ .

$$\begin{aligned} \|\nabla^2 u(t)\|_{L^6} + \|\nabla^2 p(t)\|_{L^6} &\leq \|\rho(\partial_t u + u \cdot \nabla u)\|_{L^6} \\ &\leq C(\rho_0) \left( \|\nabla \partial_t u(t)\|_{L^2} + \|(\nabla u(t))^2\|_{L^2} + \|u(t)(\nabla^2 u(t))\|_{L^2} \right) \\ &\leq C(\rho_0) \left( \|\nabla \partial_t u(t)\|_{L^2} + \|\nabla u(t)\|_{L^3} \|\nabla^2 u(t)\|_{L^2} + \|u(t)\|_{L^3} \|\nabla^2 u(t)\|_{L^6} \right) \end{aligned}$$

Applying Proposition 2.6 to the term  $\|\nabla u(t)\|_{L^3}$ , we get

$$\|\nabla^2 u(t)\|_{L^6} + \|\nabla^2 p(t)\|_{L^6} \lesssim \|\nabla \partial_t u(t)\|_{L^2} + \|\nabla u(t)\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u(t)\|_{L^2}^{\frac{3}{2}} + \|\nabla u(t)\|_{L^2} \|\nabla^2 u(t)\|_{L^6}.$$

By integration in time :

$$\begin{aligned} \int_0^t \|\nabla^2 u(t')\|_{L^6}^2 dt' + \int_0^t \|\nabla^2 p(t')\|_{L^6}^2 dt' &\lesssim \int_0^t \|\nabla \partial_t u(t')\|_{L^2}^2 dt' + \int_0^t \|\nabla u(t')\|_{L^2} \|\nabla^2 u(t')\|_{L^2}^3 dt' \\ &\quad + \|\nabla u\|_{L_T^\infty(L^2)}^2 \int_0^t \|\nabla^2 u(t')\|_{L^6}^2 dt'. \end{aligned}$$

On the one hand, Theorem 1.4 provides  $\|\nabla u\|_{L_T^\infty(L^2)}^2 \lesssim \|\nabla u_0\|_{L^2}^2$ , which implies that

$$\|\nabla u\|_{L_T^\infty(L^2)}^2 \int_0^t \|\nabla^2 u(t')\|_{L^6}^2 dt' \leq \|\nabla u_0\|_{L^2}^2 \int_0^t \|\nabla^2 u(t')\|_{L^6}^2 dt'.$$

On the other hand, applying Estimates (12) and (13) of Theorem 1.4, to the term

$$\int_0^t \|\nabla u(t')\|_{L^2} \|\nabla^2 u(t')\|_{L^2}^3 dt',$$

leads to

$$\begin{aligned} \int_0^t \|\nabla u(t')\|_{L^2} \|\nabla^2 u(t')\|_{L^2}^3 dt' &= \int_0^t \|\nabla u(t')\|_{L^2} \|\nabla^2 u(t')\|_{L^2} \|\nabla^2 u(t')\|_{L^2}^2 dt' \\ &\leq \sup_{t \in [0, T]} (\|\nabla^2 u(t)\|_{L^2}^2) \left( \int_0^t \|\nabla u(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left( \int_0^t \|\nabla^2 u(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim \|u_0\|_{L^2} \|\nabla u_0\|_{L^2} \sup_{t \in [0, T]} (\|\nabla^2 u(t)\|_{L^2}^2) \end{aligned}$$

As a result, if  $\|\nabla u_0\|_{L^2}$  is small enough, we have :

$$\begin{aligned} (90) \quad \frac{\mu}{2} \int_0^t \|\nabla^2 u(t')\|_{L^6}^2 dt' + \int_0^t \|\nabla^2 \Pi(t')\|_{L^6}^2 dt' &\lesssim \|u_0\|_{L^2} \|\nabla u_0\|_{L^2} \sup_{t \in [0, T]} (\|\nabla^2 u(t)\|_{L^2}^2) \\ &\quad + \int_0^t \|\nabla \partial_t u(t')\|_{L^2}^2 dt'. \end{aligned}$$

Summing (90) with (89) and (87), we recognize  $B_2(T)$  and we get

$$\begin{aligned} B_2(T) &\lesssim \frac{m}{2} \|\nabla u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^6 + \|u_0\|_{H^2}^2 \\ &\quad + \frac{1}{4} \int_0^t \left( \|\nabla u(t')\|_{L^2}^2 + \|\nabla^2 u(t')\|_{L^2}^2 \right) dt' \left( \sup_{t \in [0, T]} \|\nabla^2 u(t)\|_{L^2}^2 + \sup_{t \in [0, T]} \|\partial_t u(t)\|_{L^2}^2 \right) \\ &\quad + \int_0^t \|\nabla \partial_t u(t')\|_{L^2}^2 dt' + \|u_0\|_{L^2} \|\nabla u_0\|_{L^2} \sup_{t \in [0, T]} (\|\nabla^2 u(t)\|_{L^2}^2) \end{aligned}$$



The smallness condition on  $\|u_0\|_{L^2} \|\nabla u_0\|_{L^2}$  implies

$$B_2(T) \lesssim \frac{m}{2} \|\nabla u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^6 + \|u_0\|_{H^2}^2 + \frac{1}{4} \int_0^t \left( \|\nabla u(t')\|_{L^2}^2 + \|\nabla^2 u(t')\|_{L^2}^2 \right) dt' \left( \sup_{t \in [0, T]} \|\nabla^2 u(t)\|_{L^2}^2 + \sup_{t \in [0, T]} \|\partial_t u(t)\|_{L^2}^2 \right).$$

Now, we apply Gronwall lemma, and we have :

$$(91) \quad B_2(T) \lesssim \left( \frac{m}{2} \|\nabla u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^6 + \|u_0\|_{H^2}^2 \right) \exp \left( \int_0^t \|\nabla u(t')\|_{L^2}^2 dt' + \|\nabla^2 u(t')\|_{L^2}^2 dt' \right).$$

Once again Theorem 1.4 gives the expected estimate in the exponential term. Finally, we get

$$(92) \quad B_2(T) \lesssim (1 + \|u_0\|_{H^2}^4) \|u_0\|_{H^2}^2 \exp \left( \|u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2 \right).$$

This concludes the proof of 14. Up to the regularization procedure of Friedrich, we have proved the global existence of solution of 67, with data  $(\rho_0, u_0)$  satisfying hypothesis of Theorem 1.4.

## 6. APPENDIX

**Lemma 6.1.** (*Gronwall's Lemma*)

Let  $f$  and  $g$  be two positive functions satisfying  $\frac{1}{2} \frac{d}{dt} f^2(t) \leq f(t) g(t)$ . Then, we have

$$f(t) \leq f(0) + \int_0^t g(t') dt'.$$

*Proof.* We introduce the function  $H(t) \stackrel{\text{def}}{=} 2 \int_0^t f(t') g(t') dt'$ . As defined, we get immediately

$$(93) \quad H'(t) = 2 f(t) g(t) \quad \text{and} \quad f^2(t) - f^2(0) \leq H(t).$$

This implies that for any  $\varepsilon > 0$ ,

$$f(t) \leq \sqrt{H(t) + f^2(0) + \varepsilon^2}.$$

Moreover, we have in particular  $H'(t) \leq 2 \sqrt{H(t) + f^2(0) + \varepsilon^2} g(t)$  and thus

$$\frac{d}{dt} \sqrt{H(t) + f^2(0) + \varepsilon^2} \leq g(t).$$

By integration in time, we have

$$\sqrt{H(t) + f^2(0) + \varepsilon^2} \leq \sqrt{H(0) + f^2(0) + \varepsilon^2} + \int_0^t g(t') dt'.$$

Finally, we have for any  $\varepsilon > 0$ ,

$$f(t) \leq \sqrt{f^2(0) + \varepsilon^2} + \int_0^t g(t') dt',$$

which proves the result. □

**Lemma 6.2.** *The following properties hold*

(1) *Sobolev embedding: if  $p_1 \leq p_2$  and  $r_1 \leq r_2$ , then*

$$B_{p_1, r_1}^s \hookrightarrow B_{p_2, r_2}^{s - N(\frac{1}{p_1} - \frac{1}{p_2})}.$$

(2) *Product laws in Besov spaces: let  $1 \leq r, p, p_1, p_2 \leq +\infty$ .*

*If  $s_1, s_2 < \frac{N}{p}$  and  $s_1 + s_2 + N \min(0, 1 - \frac{2}{p}) > 0$ , then*

$$\|uv\|_{B_{p, r}^{s_1 + s_2 - \frac{N}{p}}} \leq C \|u\|_{B_{p, r}^{s_1}} \|v\|_{B_{p, \infty}^{s_2}}.$$

(3) Another product law: if  $|s| < \frac{N}{p}$ , then

$$\|uv\|_{B_{p,r}^s} \leq C \|u\|_{B_{p,r}^s} \|v\|_{B_{p,\infty}^{\frac{N}{p}} \cap L^\infty}.$$

(4) Algebraic properties: for  $s > 0$ ,  $B_{p,\infty}^{\frac{N}{p}} \cap L^\infty$  is an algebra. Moreover, for any  $p \in [1, +\infty)$ , then

$$B_{p,1}^{\frac{N}{p}} \hookrightarrow B_{p,\infty}^{\frac{N}{p}} \cap L^\infty.$$

**Lemma 6.3.** Let  $\mathcal{C}$  a ring of  $\mathbb{R}^3$ . A constant  $C$  exists so that for any positive real number  $\lambda$ , any non-negative integer  $k$ , the following hold

$$\text{If } \text{Supp } \hat{u} \subset \lambda \mathcal{C}, \text{ then } C^{-1-k} \lambda^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^{1+k} \lambda^k \|u\|_{L^a}.$$

**Lemma 6.4.**

$$(94) \quad \sum_{q \in \mathbb{Z}} 2^{\frac{3q}{2}} \|[\Delta_q, S_m a] \nabla u\|_{L_t^1(L^2)} \leq C \left( 2^m \|a\|_{L_t^\infty(L^\infty)} \|u\|_{L_t^1(B_{2,1}^{\frac{3}{2}})} + 2^{2m} \|a\|_{L_t^\infty(L^2)} \|u\|_{L_t^1(\dot{H}^2)} \right)$$

*Proof.* By virtue of Bony's decomposition, the commutator may be decomposed into

$$(95) \quad \begin{aligned} [\Delta_q, S_m a] &= \Delta_q(S_m a \nabla u) - S_m a \Delta_q \nabla u \\ &= \Delta_q(T_{S_m a} \nabla u) + \Delta_q(T_{\nabla u} S_m a) + \Delta_q R(S_m a, \nabla u) \\ &\quad - T_{S_m a} \Delta_q \nabla u - T_{\Delta_q \nabla u} S_m a - R(S_m a, \Delta_q \nabla u) \\ &= [\Delta_q, T_{S_m a}] \nabla u + \Delta_q(T_{\nabla u} S_m a) + \Delta_q R(S_m a, \nabla u) - T'_{\Delta_q \nabla u} S_m a, \end{aligned}$$

where  $T'_a b \stackrel{\text{def}}{=} T_a b + R(a, b)$ . Let us analyse each term in the right-hand-side. Firstly, we decompose the first commutator term into

$$(96) \quad \begin{aligned} [\Delta_q, T_{S_m a}] \nabla u &= \Delta_q(T_{S_m a} \nabla u) - T_{S_m a} \Delta_q \nabla u \\ &= \Delta_q \left( \sum_{|q-q'| \leq 4} S_{q'-1} S_m a \Delta_{q'} \nabla u \right) - \sum_{|q-q'| \leq 4} S_{q'-1} S_m a \Delta_{q'} \Delta_q \nabla u \\ &= \sum_{|q-q'| \leq 4} [\Delta_q, S_{q'-1} S_m a] \Delta_{q'} \nabla u. \end{aligned}$$

Now, let us focus on the commutator term  $[\Delta_q, S_{q'-1} S_m a] \Delta_{q'} \nabla u$ . We shall use definition of Littlewood-Paley theory.

$$(97) \quad \begin{aligned} [\Delta_q, S_{q'-1} S_m a] \Delta_{q'} \nabla u &= \Delta_q \left( S_{q'-1} S_m a \Delta_{q'} \nabla u \right) - S_{q'-1} S_m a \Delta_q \Delta_{q'} \nabla u \\ &= \varphi(2^{-q} |D|) S_{q'-1} S_m a \Delta_{q'} \nabla u - S_{q'-1} S_m a \varphi(2^{-q} |D|) \Delta_{q'} \nabla u. \end{aligned}$$

In particular, writting  $h \stackrel{\text{def}}{=} \mathcal{F}^{-1} \varphi(|\cdot|)$ , we get

$$(98) \quad \begin{aligned} \varphi(2^{-q} |D|) S_{q'-1} S_m a \Delta_{q'} \nabla u(x) &\stackrel{\text{def}}{=} \int_{\mathbb{T}^3} 2^{qd} h(2^q y) S_{q'-1} S_m a(x-y) \Delta_{q'} \nabla u(x-y) dy \\ &= \int_{\mathbb{T}^3} h(z) S_{q'-1} S_m a(x-2^{-q} z) \Delta_{q'} \nabla u(x-2^{-q} z) dz. \end{aligned}$$

Likewise, we have

$$(99) \quad \begin{aligned} S_{q'-1} S_m a \varphi(2^{-q} |D|) \Delta_{q'} \nabla u(x) &\stackrel{\text{def}}{=} S_{q'-1} S_m a(x) \int_{\mathbb{T}^3} 2^{qd} h(2^q y) \Delta_{q'} \nabla u(x-y) dy \\ &= \int_{\mathbb{T}^3} S_{q'-1} S_m a(x) h(z) \Delta_{q'} \nabla u(x-2^{-q} z) dz. \end{aligned}$$

Therefore, applying the first-order Taylor's formula, we get, for any  $x \in \mathbb{T}^3$ ,

(100)

$$\begin{aligned} [\Delta_q, S_{q'-1} S_m a] \Delta_{q'} \nabla u(x) &= \int_{\mathbb{T}^3} h(z) [S_{q'-1} S_m a(x - 2^{-q} z) - S_{q'-1} S_m a(x)] \Delta_{q'} \nabla u(x - 2^{-q} z) dz \\ &= - \int_{\mathbb{T}^3} \int_0^1 h(z) 2^{-q} z \cdot \nabla S_{q'-1} S_m a(x - 2^{-q} z t) \Delta_{q'} \nabla u(x - 2^{-q} z) dz dt \\ &= -2^{-q} \int_{\mathbb{T}^3} \int_0^1 2^{qd} h(2^q y) y \cdot \nabla S_{q'-1} S_m a(x - y t) \Delta_{q'} \nabla u(x - y) dz dt. \end{aligned}$$

Therefore, we infer that, for any  $x \in \mathbb{T}^3$ ,

$$\| [\Delta_q, S_{q'-1} S_m a] \Delta_{q'} \nabla u \|_{L^2} \leq \| \nabla S_{q'-1} S_m a \|_{L^\infty} 2^{-q} \left\| \int_{\mathbb{T}^3} 2^{qd} (2^q y) h(2^q y) \Delta_{q'} \nabla u(\cdot - y) dz \right\|_{L^2}.$$

Applying Young's inequality ( $L^1 * L^2 = L^2$ ), we infer that

$$(101) \quad \| [\Delta_q, S_{q'-1} S_m a] \Delta_{q'} \nabla u \|_{L^2} \leq C \| \nabla S_{q'-1} S_m a \|_{L^\infty} 2^{-q} \| \Delta_{q'} \nabla u \|_{L^2}.$$

Obviously, we have

$$\| \nabla S_{q'-1} S_m a \|_{L^\infty} \leq \| \nabla S_m a \|_{L^\infty} \leq 2^m \| a \|_{L^\infty}.$$

Finally, we get

$$\| [\Delta_q, S_{q'-1} S_m a] \Delta_{q'} \nabla u \|_{L^2} \leq C 2^{-q} 2^m \| a \|_{L^\infty} \| \Delta_{q'} \nabla u \|_{L^2},$$

and thus,

$$(102) \quad \| [\Delta_q, T_{S_m a}] \nabla u \|_{L^2} \leq C \sum_{|q-q'| \leq 4} 2^{-q} 2^m \| a \|_{L^\infty} \| \Delta_{q'} \nabla u \|_{L^2}.$$

As a consequence, we have

$$\begin{aligned} (103) \quad 2^{\frac{3q}{2}} \| [\Delta_q, T_{S_m a}] \nabla u \|_{L^2} &\leq C \sum_{|q-q'| \leq 4} 2^{\frac{3q}{2}} 2^{-q} 2^m \| a \|_{L^\infty} 2^{\frac{-q'}{2}} 2^{\frac{q'}{2}} \| \Delta_{q'} \nabla u \|_{L^2} \\ &\leq C 2^m \| a \|_{L^\infty} \sum_{|q-q'| \leq 4} 2^{\frac{q-q'}{2}} 2^{\frac{q'}{2}} \| \Delta_{q'} \nabla u \|_{L^2}. \end{aligned}$$

By definition of the Besov norm, there exists a serie  $(c_{q'})_{q \in \mathbb{Z}}$  belonging to  $\ell^1(\mathbb{Z})$  such that

$$2^{\frac{q'}{2}} \| \Delta_{q'} \nabla u \|_{L^2} \leq C c_{q'} \| \nabla u \|_{B_{2,1}^{\frac{1}{2}}}.$$

And thus,

$$(104) \quad 2^{\frac{3q}{2}} \| [\Delta_q, T_{S_m a}] \nabla u \|_{L^2} \leq C 2^m \| a \|_{L^\infty} \| \nabla u \|_{B_{2,1}^{\frac{1}{2}}} \sum_{|q-q'| \leq 4} 2^{\frac{q-q'}{2}} c_{q'}.$$

We notice, by virtue of Young's inequality, that the term  $\sum_{|q-q'| \leq 4} 2^{\frac{q-q'}{2}} c_{q'}$  belongs to  $\ell^1(\mathbb{Z})$ . Indeed, let

us define  $d_q \stackrel{\text{def}}{=} \sum_{|q-q'| \leq 4} 2^{\frac{q-q'}{2}} c_{q'}$ . Thanks to Young's inequality, we get

$$\| d_q \|_{\ell^1(\mathbb{Z})} \leq \| c_q \|_{\ell^1(\mathbb{Z})} \times \sum_{-4 \leq k \leq 4} 2^{\frac{k}{2}} \leq C.$$

Finally, we get

$$(105) \quad \begin{aligned} \sum_{q \in \mathbb{Z}} 2^{\frac{3q}{2}} \|\Delta_q T_{S_m a} \nabla u\|_{L^2} &\leq C 2^m \|a\|_{L^\infty} \|\nabla u\|_{B_{2,1}^{\frac{1}{2}}} \sum_{q \in \mathbb{Z}} d_q \\ &\leq C 2^m \|a\|_{L^\infty} \|\nabla u\|_{B_{2,1}^{\frac{1}{2}}}. \end{aligned}$$

By integration in time, we infer that

$$(106) \quad \sum_{q \in \mathbb{Z}} 2^{\frac{3q}{2}} \|\Delta_q T_{S_m a} \nabla u\|_{L_t^1(L^2)} \leq C 2^m \|a\|_{L_t^\infty(L^\infty)} \|\nabla u\|_{L_t^1(B_{2,1}^{\frac{1}{2}})}.$$

This gives the first term in the Lemma. The second term will stem from remainder terms in the Bony's decomposition. More precisely, concerning the term  $\sum_{q \in \mathbb{Z}} 2^{\frac{3q}{2}} \Delta_q T_{\nabla u} S_m a\|_{L_t^1(L^2)}$ , we have by definition

$$\sum_{q \in \mathbb{Z}} 2^{\frac{3q}{2}} \|\Delta_q T_{\nabla u} S_m a\|_{L_t^1(L^2)} \stackrel{\text{def}}{=} \|T_{\nabla u} S_m a\|_{B_{2,1}^{\frac{3}{2}}}.$$

By virtue of Theorem 2.82 in the book [5], we have

$$(107) \quad \|T_{\nabla u} S_m a\|_{B_{2,1}^{\frac{3}{2}}} \leq C \|\nabla u\|_{B_{\infty,2}^{-\frac{1}{2}}} \|S_m a\|_{B_{2,2}^2}.$$

Moreover, Bernstein result implies the following embedding  $B_{2,2}^1 \hookrightarrow B_{\infty,2}^{-\frac{1}{2}}$ . Therefore, we have

$$(108) \quad \|T_{\nabla u} S_m a\|_{B_{2,1}^{\frac{3}{2}}} \leq C \|\nabla u\|_{B_{2,2}^1 \equiv H^1} \|S_m a\|_{B_{2,2}^2}.$$

Applying Poincaré-Wirtinger to  $\|\nabla u\|_{H^1}$ , (since the average of  $\nabla u$  is nul), we infer that the norms  $\|\nabla u\|_{H^1}$  and  $\|\nabla u\|_{\dot{H}^1}$  are equivalent and thus

$$(109) \quad \|T_{\nabla u} S_m a\|_{B_{2,1}^{\frac{3}{2}}} \leq C \|u\|_{\dot{H}^2} \|S_m a\|_{B_{2,2}^2}.$$

On the other hand, it seems obvious that  $\|S_m a\|_{B_{2,2}^2} \leq \|S_m a\|_{\dot{B}_{2,2}^2} \leq \|S_m a\|_{\dot{H}^2}$ . As a result,

$$(110) \quad \|T_{\nabla u} S_m a\|_{B_{2,1}^{\frac{3}{2}}} \leq C 2^{2m} \|u\|_{\dot{H}^2} \|S_m a\|_{\dot{H}^2}.$$

Finally, by integration in time and by definition of  $S_m a$ , we get

$$(111) \quad \|T_{\nabla u} S_m a\|_{L_t^1(B_{2,1}^{\frac{3}{2}})} \stackrel{\text{def}}{=} \sum_{q \in \mathbb{Z}} 2^{\frac{3q}{2}} \|\Delta_q T_{\nabla u} S_m a\|_{L_t^1(L^2)} \leq C 2^{2m} \|u\|_{L_t^1(\dot{H}^2)} \|a\|_{L_t^\infty(L^2)}.$$

The estimate on the term  $\sum_{q \in \mathbb{Z}} 2^{\frac{3q}{2}} \|\Delta_q R(S_m a, \nabla u)\|_{L_t^1(L^2)}$  is close to the previous one, by virtue of Theorem page 2.85 in [5]. We recall it below.

*Remind: If  $s_1$  and  $s_2$  are two real numbers, such that  $s_1 + s_2 > 0$ , then*

$$\|R(u, v)\|_{B_{p,r}^{s_1+s_2}} \leq C(s_1, s_2) \|u\|_{B_{p_1,r_1}^{s_1}} \|v\|_{B_{p_2,r_2}^{s_2}}, \quad \frac{1}{p} \stackrel{\text{def}}{=} \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{r} \stackrel{\text{def}}{=} \frac{1}{r_1} + \frac{1}{r_2}.$$

Therefore, we have

$$(112) \quad \sum_{q \in \mathbb{Z}} 2^{\frac{3q}{2}} \|\Delta_q R(S_m a, \nabla u)\|_{L_t^1(L^2)} \stackrel{\text{def}}{=} \|R(S_m a, \nabla u)\|_{L_t^1(B_{2,1}^{\frac{3}{2}})} \leq C 2^{2m} \|u\|_{L_t^1(\dot{H}^2)} \|a\|_{L_t^\infty(L^2)}.$$

Concerning the last term,  $\sum_{q \in \mathbb{Z}} 2^{\frac{3q}{2}} \|T'_{\Delta_q \nabla u} S_m a\|_{L_t^1(L^2)}$ , we write the definition. Indeed,

$$T'_{\Delta_q \nabla u} S_m a \stackrel{\text{def}}{=} \sum_{q' \geq q-2} S_{q'+2} \Delta_q \nabla u \Delta_{q'} S_m a.$$

Therefore, we get

$$\begin{aligned}
 \|T'_{\Delta_q \nabla u} S_m a\|_{L^2} &\leq C \sum_{q' \geq q-2} \|\Delta_q \nabla u\|_{L^\infty} \|\Delta_{q'} S_m a\|_{L^2} \\
 (113) \quad 2^{\frac{3q}{2}} \|T'_{\Delta_q \nabla u} S_m a\|_{L^2} &\leq C 2^{\frac{3q}{2}} \sum_{q' \geq q-2} 2^{\frac{q}{2}} 2^{-\frac{q}{2}} \|\Delta_q \nabla u\|_{L^\infty} 2^{-2q'} 2^{2q'} \|\Delta_{q'} S_m a\|_{L^2} \\
 &\leq C \sum_{q' \geq q-2} 2^{2(q-q')} 2^{-\frac{q}{2}} \|\Delta_q \nabla u\|_{L^\infty} 2^{2q'} \|\Delta_{q'} S_m a\|_{L^2}
 \end{aligned}$$

By definition of the Besov norm, there exists a sequence  $c_{q'}$  belonging to  $\ell^2(\mathbb{Z})$  such that

$$2^{2q'} \|\Delta_{q'} S_m a\|_{L^2} \leq C c_{q'} \|S_m a\|_{B_{2,2}^2}.$$

As a result, by summation on  $q$ , we infer that

$$(114) \quad \sum_{q \in \mathbb{Z}} 2^{\frac{3q}{2}} \|T'_{\Delta_q \nabla u} S_m a\|_{L^2} \leq C \left( \sum_{q \in \mathbb{Z}} 2^{-\frac{q}{2}} \|\Delta_q \nabla u\|_{L^\infty} d_q \right) \|S_m a\|_{B_{2,2}^2},$$

where the sequence  $d_q$  stems from convolution product:  $d_q \stackrel{\text{def}}{=} \sum_{q' \geq q-2} 2^{2(q-q')} c_{q'}$ . As defined, it is clear that, by virtue of Young's inequality,  $\|d_q\|_{\ell^2(\mathbb{Z})} \leq C$ . Finally, Cauchy-Schwarz inequality yields

$$(115) \quad \sum_{q \in \mathbb{Z}} 2^{\frac{3q}{2}} \|T'_{\Delta_q \nabla u} S_m a\|_{L^2} \leq C \|\nabla u\|_{B_{\infty,2}^{-\frac{1}{2}}} \|S_m a\|_{B_{2,2}^2},$$

Once again, the Bernstein's embedding  $B_{2,2}^1 \hookrightarrow B_{\infty,2}^{-\frac{1}{2}}$ , combining with an integration in time gives

$$(116) \quad \sum_{q \in \mathbb{Z}} 2^{\frac{3q}{2}} \|T'_{\Delta_q \nabla u} S_m a\|_{L_t^1(L^2)} \leq C \|S_m a\|_{L_t^\infty(B_{2,2}^2)} \|\nabla u\|_{L_t^1(B_{2,2}^1)}$$

Therefore,

$$(117) \quad \sum_{q \in \mathbb{Z}} 2^{\frac{3q}{2}} \|T'_{\Delta_q \nabla u} S_m a\|_{L_t^1(L^2)} \leq C 2^{2m} \|a\|_{L_t^\infty(L^2)} \|u\|_{L_t^1(\dot{H}^2)}$$

**Conclusion** Summing estimates (106), (111), (112), and (117) completes the proof of the Lemma.  $\square$

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